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Report on the Professorial thesis of Xavier Mary

Generalized inverses, semigroups and rings

It is a pleasure to be able to read and recommend the thesis of Mary for the award of Habilitation à diriger des Recherches. I have had personal acquaintance with Dr Mary for over 5 years, and was aware of his work for some time before that. As I explain below, Mary offers unique insights into a number of algebraic notions, and the thesis demonstrates his significant contributions to several areas. Part IV of the thesis is entitled **Algebraic theory of semigroups and structure theorems** and it is this part I have been asked to review. It is broken into 6 chapters, and I comment briefly on their content below, before returning to an overall assessment.

Chapter 13: The inverse along an element in semigroups As indicated earlier in the thesis, a number of different definitions of the inverse of an element in a semigroup have been proposed. A breakthrough by Mary came in [146], where he defines the notion of the *inverse of an element a along an element d* (in fact, it is the $\mathcal{H}$-class of d that is significant). The chapter reviews this extremely neat and useful notion, giving its main properties, connecting it with the work of Costa and Steinberg on Schützenberger categories, and considering it in the light of classical work of Miller and Clifford. This is all very pretty and at some points quite suprising, highlighting some previously undiscovered aspects of regular $\mathcal{D}$-classes in semigroups. Mary also gives a cogent explanation of how his work fits with a concurrent and similar approach of Drazin.

Chapter 14: Classes of semigroups modulo Green's relation $\mathcal{H}$ The aim here is to take various natural semigroup definitions and replace them with the corresponding definition 'modulo $\mathcal{H}$'. The main basis for this work is [148]. Mary is motivated by the fact that an element x is invertible along an element a (as mentioned above) if and only if $axa\,\mathcal{H}\,a$. In this case, he says that x is an *inner inverse of a modulo $\mathcal{H}$*. In this way he connects bands with completely regular semigroups, semilattices with Clifford semigroups, and orthodox semigroups with both $\mathcal{H}$-orthodox and E-solid semigroups. (Classes such as that of orthodox semigroups may have alternative definitions, and applying the adaptation modulo $\mathcal{H}$ to these different definitions may result in different classes). Some useful diagrams are provided. In addition, Mary explains how his work relates to that of Petrich [192]. The fact that Mary's approaches and ideas dovetail with those of others I find a strength, rather than a weakness, since it indicates that his ideas have a very natural basis.

Chapter 15: Completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups A semigroup is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant for a subset E of idempotents if every $\widetilde{\mathcal{H}}_E$-class contains an idempotent of E, and certain compatibility conditions hold. Such semigroups are a very natural extension of the well-studied class of completely regular semigroups, essentially obtained by replacing Green's relations by 'generalised' Green's relations. The main source for this chapter is [150]. I pick out here two main strands. One is the structure of a completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup (as a semilattice of semigroups of the same kind, but satisfying the appropriate simplicity condition). The second is the characterisation of a number of classes of completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups as varieties of algebras of signature $(2, 1)$. Here, the additional unary operation picks out the element a^+ of E in the same $\widetilde{\mathcal{H}}_E$-class of a given element a. I am aware of a very broad interest amongst semigroup theorists in these ideas, from both Eastern and Western traditions. To pin down the exact conditions needed to move away from the regular case to this more general situation is not always easy, and Mary has succeeded well in this regard.

Chapter 16: Chains of associate idempotents and chained semigroups The theme here is closely associated with the current hot topic of biordered sets. Although the catalyst was using chains of idempotents in endomorphism rings to study *modules*, the notion of an idempotent chain is already well known to semigroup theorists, arising from the seminal work of Nambooripad. In [139] Mary uses idempotent chains to refine notions of regularity. In this way he can find new characterisations of, for example, completely regular semigroups. His work can be used to deduce earlier structure theorems for π-regular semigroups from [20].

Chapter 17: Semigroup biacts The focus of this chapter is the work of [152]. Here, Mary introduces the reader to semigroup biacts and then explains how the classic Green's relations for semigroups can be adapted to biacts, and that (somewhat surprisingly) some of their pleasing properties translate to this context, in particular, $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. The chapter goes on to give two versions of Mary's structure theorem for faithful, stable, $\mathcal{J}$-simple biacts. The chapter concludes by pointing out that Mary's results have important consequences, enabling one to deduce existing results by Oehmke and Steinberg. These techniques turn out to be wide-ranging, impinging on both analogues of the Kaloujnine-Krasner theorem, and Krohn-Rhodes type decomposition theorems.

Chapter 18: Partial orders on arbitrary (non-regular) semigroups This short chapter reviews existing notions of partial orders on semigroups, inspired by that for inverse semigroups. The latter may be determined by idempotents, and, owing to the natural embedding of an inverse semigroup into a symmetric inverse semigroup, may be viewed as restriction of mappings. Existing notions of partial orders may not transfer to non-regular semigroups, and this is where the work of Mary comes in. The articles [78] and [79] use outer inverses to show how to approach the definition of partial orders in such a way they can be lifted to the context of non-regular semigroups. This requires some ingenuity and insight.

I have made some comments on the quality of Mary's work whilst outlining the contents of the chapters in Part IV of the thesis: let me now enlarge. It is remarkable that Mary has made contributions to so many different areas, and for some of them (certainly semigroup theory and the theory of acts) has come into the area at a relatively mature stage. This is not easy to do. But, Mary has unique and off-beat ideas and has brought fresh perspectives to semigroup theory. The main thrust of his work in this direction is to find tools to study semigroups that do not have to be (von Neumann) regular. Many of the methods of classical semigroup theory (such as Green's relations) are not of great use here, but Mary has been innovative and wide ranging in his approaches. I would highlight here his work on acts in Chapter 17, with its extensive consequences, and also the notion of studying properties of semigroups 'modulo $\mathcal{H}$' in Chapter 14. The potential block to the latter (which I would have thought a serious one) is that $\mathcal{H}$ does not have to be a congruence on a general semigroup, but Mary has shown that this is not necessary to get his techniques to work. I am also impressed by the way that Mary relates his work to so many different strands of the existing semigroup literature, showing a deal of scholarliness as he does so.

To summarise, I have been enjoying following Mary's work over the last decade or so. Reading the Professorial thesis only confirms to me the range and quality of his work. I thoroughly recommend the award of Habilitation á diriger des Recherches.

Sincerely yours

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