Report on Xavier Mary's Habilitation Thesis

BY André Leroy

I never met Xavier Mary face to face but we met online, in particular when he once gave a very interesting talk about perspectivity in our seminar in Lens. I was by this time interested in the question of separativity of regular rings and this subject is strongly related to perspectivity. So I was happy to listen to him and really enjoyed his conference. In fact, the question of separativity of regular is still unsolved (after 40 years) and I have some hope that the techniques developed by Xavier Mary and his collaborators could help solving this question. I personally knows two of his collaborators Pace Nielsen and Dinesh Khurana. They are internationally very well recognized as talented ring theorists.

This thesis is remarkable by its unity and its diversity! The unity is through the "elementwize" approach the diversity is due to the fact that it concerns semigroup, rings and modules. Knowing that Dr Xavier Mary originally worked in functional analysis immediately gives the feeling that the Author has very wide knowledge. This is confirmed all along the work specially when he gives his global point of view on a subject with clear overview and up to date information. Let me mention that this "elementwize" point of view leads to more structural important concepts such as decomposition of objects and cancellation properties. This is in particular developed along the part IV of this work (eseentially for semigroups) but also in part V (ring theory, see below). This connection is very often due to the study of idempotent elements (hence to regular elements and their many other associated kind of elements). Another feature of the text is that it is very much self contained in the sense that most of the basic definitions and the key results are given. Unpublished results are offered and are clearly marked as such.

Before going into a description of the results that most interested me, let me mention that the "elementwize" point of view which is developped all along the thesis is far from new even if it is intensively studied for its own sake only quite recently. Of course, idempotent, invertible and nilpotent elements are unavoidable but elements such as regular, unit regular, exchange, clean, strongly regular,...are in fact also very natural and are present for about 50 years. They form nowadays a fascinating and extending subject and Xavier Mary is certainly one of the best specialist in this area.

Starting with semigroup point of view on questions that I would put under the name of "factorization problem", Dr. Xavier Mary first present the Green relations and the classical Miller and Clifford's theorem. One of the most important new notion is then introduced "the inverse along an element". This is done in the semigroup frame but has many important impacts in the ring theory as well. The inverse along idempotents play indeed a very important role since it is in particular an inverse in the corner semigroup defined by the idempotent. The analogue of (Kholia-)Drazin inverse is introduced and the Cline formula for inverses along some idempotents are presented.

The first thing to notice is that the existence of inverses along an element creates (genuine) inverses and (special) clean elements. The case of triangular matrix over Dedekind finite ring is explored in details with complete formulas for the inverses along an element (the case of a general 2×2 matrix with inverse along a triangular matrix is also explicitly computed). The Jacobson Lemma is presented even in the case of one sided invertible elements or in the case of non unital rings (with the help of the circle operation). The absorption law is also given for inverses along an element. The two short but interesting chapters 9 and 10 presents the group inverse of a product in rings. I am not a specialist but I am honestly quite surprised that this was not known until recently discovered by Xavier Mary and P. Patricio.

As a ring theorist I'll now concentrate, on part V. This part starts with a set of "classical definitions" and their known connections. In fact, these relations are far from well known and it is a pleasure to see them all in one place. The first sections give characterizations of different elements such as exchange, clean, strongly clean and special clean via the use of generalized outer inverses. Let me mention a few of the results: Let $a \in R$ and assume that a is invertible along e, $1-a$ is invertible along $1-e$ for some idempotent e Then a is clean.

Let $a \in R$. Then a is strongly clean if and only if there exists $x, y \in R$ such that:

- $xax = x, ax = xa;$
- $y(1-a)y = y, ay = ya;$
- $(1-a)y = 1 - ax.$

The following result gives characterization of strongly clean elements:

Let $a \in R$. Then the following statements are equivalent:

(1) a is strongly regular;
(2) There exist a clean decomposition $a = e + u$ where $e = e^2$, $u \in R^{-1}$ such that $ae = 0$;
(3) There exist a strongly clean decomposition $a = e + u$ with $e = e^2$ and u is invertible such that $ae = 0 = ea$.

Xavier Mary also proved that there is a bijective correspondence between special clean decompositions and strongly regular reflexive inverses.

The next chapter gives a complete study of Cline's formula and the Jacobson lemma. These are classical themes but many strong generalizations are propsosed in this work. Classically the Jacobson lemma says that in a ring, if $1 + ab$ is invertible then, so is $1 + ba$ is also invertible $(1+ba)^{-1} = 1 - b(1+ab)^{-1}a$. This chapter starts with the reverse order law (the inverse of a product is the inverse of the factors in reverse order) and states that under the assumption that the ring is Dedekind finite the reverse order law is valid for group invertible elements. An example shows that the Dedekind assumption is not superfluous. The Cline formula is given for bicommuting weak inverses and is extended further via a lattice isomorphism by working on the circle ring (the ring $(R, \oplus, \circ)$, where $x \oplus y = x + y - 1$ and $x \circ y = x + y - xy$)

The next chapter concerns unit regular elements in a ring without unity. Different possibilities are explored along the thesis but let me just mention the Q-regularity which involves the quasi regular element as a susbstitute as follows: an element a is unit regular (or q-unit regular) if there exists a nonzero Q-regular element q such that $a = a^2 + aqa$. The (general) rings R that are Q unit regular are characterized as the ones such that any unitarization leads to a unit regular ring or equivalently as the ring such that the Dorroh extension is unit regular or equivalently as the ring such that isomorphic idempotent have isomorphic complements. Many other characterizations are given, as well as an unpublished result which says that the general ring R is Q unit regular if and only if isomorphic idempotent are also isomorphic in the circle ring. The next kinds of elements analyzed are the concept of exchange, clean and special clean elements in nonunital rings. The concepts were introduced by P. Ara and W.K. Nicholson and are quite natural replacing invertible elements by nonzero regular ones. Many characterizations of the new definitions are given and placed in the general frame giving a good general view of all the interrelation of these elements both in the general situation and in the unital one. This chapter ends with a remarkable version of the Jacobson theorem in the case of a general ring.

The next chapter is concerned with special clean and perspective elements. The special clean elements appeared in a paper by V. Camillo and D. Khurana where they showed that unit regular rings are clean. The first question addressed by Dr. Mary is the following: Is there a simple criterion to decide whether an element a is special clean? The method to answer that question followed by Xavier Mary and his collaborators is to search, among the units, the inner inverses $u-1$ of a such that $a-u$ is an idempotent. The tools in this quest are Morita context (induced by an idempotent via Pierce decomposition) and what is called the Schur complement of a matrix in a Morita context. The key tool to answer the question of special cleaness of a is a remarkable equation whose solutions give the set of special clean decompositions of a. Many examples are given showing the wide range of these results. From them, one can recover in an element-wise manner some classical theorems. In particular, the fact that

- a ring is unit-regular if and only if it is special clean;
- an exchange ring has stable range one if and only if its regular elements are unit-regular;
- if idempotents are central modulo the Jacobson radical $J(R)$, then an element is regular iff it is strongly regular.

Perspectivity of rings is an important and well studied subject in module theory. In fact, this notion comes from lattice theory. Perspectivity in modules traces back J. Von Neumann and his studies on continuous geometries. It has then been reconsidered in the 60's and 70's by L. Fuchs and D. Handelman, in link with cancellation and substitution properties. In this thesis it is extended to elements. First we recall the classical definition for modules: Let R be a ring, M be a (right) module and $A, A' \subseteq_{\oplus} M$ be direct summands of M, then A and A' are perspective if they share a common complementary summand. Now, an element a of a ring R is right perspective if it is von Neumann regular and such that every complementary summand of its right annihilator $r_R(a)$ is perspective with aR. In fact this notion is actually left right symmetric. Different characterizations of perspective elements are given. The first objective of Dr. Mary is to relate perspective elements with clean and special clean decompositions and to show tight connections between these. Some of them are a bit technical and don't really fit in a report. Let me rather give two of their nice consequences.

1) Let R be a ring, and a a regular element. If $a^2 = 0$, then a is perspective.

2) Let a be a unit regular element of a ring R with unit inner inverse v^{-1} and let $f = av^{-1}$. Under any of the following conditions, a is perspective:

(1) $(1-f)R(1-f)$ has stable range 1;
(2) fRf has stable range 1;
(3) $(1-f)RfR(1-f) \subseteq J(R)$.

A small example is given showing that an element can be unit-regular and clean and hence special clean without being perspective.

The last chapter concerns the important and ubiquitous notion of chain of idempotents. To present the chains we need the following definition: idempotents e and f of a ring R are left (resp. right) associates if $Re = Rf$ (resp. $eR = fR$). This is denoted by $e \sim_l f$ (resp. $e \sim_r f$) (Green's relation $\mathcal{L}$ restricted to the subset $E(R)$ of idempotents). The following properties are defined:

$\mathcal{P}(n)$: R is (strongly) n-chained if any two isomorphic idempotents are connected by both a left and a right n-chain.

$\mathcal{D}(n)$: Any two conjugate idempotents are connected by a left and a right (equiv. only a left) association chain of length n;

R is weakly $\mathcal{P}(n)$ if any two isomorphic (resp. conjugate) idempotents are connected by either a left and a right association chain of length n.

Strong and weak terminology is also used elementwise.

If A is a direct summand of a module M, we denote by $\overline{A}$ any complementary summand of A. The direct summands $A, A' \subseteq_{\oplus}$ are 0-perspective if $A = A'$, (denoted $A \sim \oplus^0 A'$). Then, for any $p \in \mathbb{N}, A, A'$ are $p+1$-perspective (denoted $A \sim_{\oplus} p+1 A'$) if $A \sim_{\oplus}^{p} B \sim_{\oplus} A'$ for some direct summand B

The module M is $p+1/2$-perspective, if whenever $M = A \oplus \overline{A}$ and $A \simeq A'$ for some direct summands A, A' then $M = A' \oplus \overline{A'}$ for some $\overline{A'} \subseteq_{\oplus} M$ such that $\overline{A} \sim_{\oplus}^{p} \overline{A'}$.

These definitions applied to endomrphisms of a module M when $im(a)$, ker(a) are direct summands.

It is quite difficult to give precise statement of the many results given in the thesis, so I'll just give a few of the one that I like most.

For a ring R, the following are equivalent:

(1) Any two isomorphic idempotents are strongly 2-chained;

(2) $reg(R) = sreg(R)$;

(3) $ureg(R) = sreg(R)$;

(4) $sp.cl(R) = sreg(R)$;

(5) Any two idempotents in the same isomorphic class are weakly 2-chained;

(6) Idempotents of R are central modulo the Jacobson radical.

Equipped with the notions just introduced many characterizations of perspectivity of an element and of modules are offered in the thesis.

Let me mention the following one: Let M be a module, $R = End(M)$ and $MR = (R, .)$. Then M is perspective if and only if R_R is perspective if and only if regular elements of R are perspective if and only if the monoid MR satisfies $\mathcal{P}(3)$ if and only if it satisfies $\mathcal{P}(3)$ weakly. There are also strong connections between perspectivity of an element and "local" stable range one.

For an element in the endomorphism ring of a module Dr. Xavier Mary also gets a lot of characterizations. Let me mention the following ones:

Let M be a module, and $a \in R = End(M)$. Let also $MR = (R, .)$. Then the following statements are equivalent: (1) a is image (equiv. kernel) 3/2-perspective; (2) a is 4-chained regular; (3) aR and bR are perspective, for some $b \in V(a)$ (the set of reflexive inverses of a); (4) a has a completely regular reflexive inverse (as an element of MR); (5) a is special clean (as an element of the ring R).

As a last topic that I'll mention in this short report, let me go to the transitivity of perspectivity.

For a regular ring R, the following conditions are well known to be equivalent:

(2) The ring R is IC (internal cancellation holds)

(3) The ring R is perspective ring.
(4) The ring $M_2(R)$ has transitivity of perspectivity.
(5) The ring R has stable range one.

These equivalenes are not true for a ring R that is not regular. The question quickly arises that any IC ring with transitive perspectivity could be perspective but a counter example is given.

Another intersting result is that a ring R is such that weak 2-chaining is transitive, then idempotents are central modulo the Jacobson radical.

Let us also mention the following facts: let $n \geq 2$, assume that R is a projective-free ring with n in its stable range. If $m \geq 4n - 5$, then $S = M_m(R)$ satisfies $P(4)$. If R does not have 1 in its stable range, then $P(3)$ fails (i.e., S is not a perspective ring)

The paper ends with questions related to the results presented in this part 5. Looking at all the material that is presented in this thesis, and knowing that the product of idempotents is strongly related to the question of separativity of regular rings, I would be curious to look if some of the results presented could have an impact on this old question.

Let me end this report with a global evaluation. This is really a great work and I am glad to have all these results cleraly stated in one place. The thesis is very carefully divided. every chapter ends with interesting questions. I really hope to be able to look at some of them. My deep feeling is that the notions introduced and studied by Dr. Xavier Mary and its collaborators will have a profound and longstanding impact on the future of semigroup theory and ring theory. My last sentence is thus that Dr. Xavier Mary is fully ready to guide students and I do hope that together with them he will continue to develop this important area.

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