Report on Habilitation Thesis of

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"**Generalized inverses, semigroups and rings**"

From the middle of the last century, generalized inverses of matrices play an important role in theoretical and numerical methods of linear algebra and have numerous applications in statistics, econometrics, logistics, electrical network theory, the theory of differential and difference equations. In the cases when ordinary inverses of matrices do not exist generalized inverses are used to solve some matrix equations. Similar reasoning can be applied to linear (bounded or unbounded) operators on Banach and Hilbert spaces, elements of Banach and C*-algebras, or more generally, in rings with or without involution and even semigroups. Various kinds of partial orders have been investigated in terms of generalized inverses on semigroups, rings, matrices or operators. Partial orders of matrices that may be derived from generalized inverses have applications in the theory of linear statistical models.

This habilitation thesis contains contributions of author (with or without co-authors). Beside the author's main mathematical achievements, this memoir presents personal comments which can help to understand the notions, considering problem, seminal results used in author's research and recent progress on a close topic by others mathematicians. In addition, open problems for future works are stated at the end of parts. Part II and Part III of this thesis are dedicated to generalized inverses.

In Part I, the basic and necessary definitions, notations and properties of generalized inverses, semigroups and rings are introduced. Also, the author explains the problems that inspired him and results that he get.

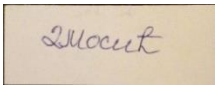
The very significant contribution of the author to the theory of generalized inverses is the introduction of the inverse along an element that encompass the well-known generalized inverses as the group inverse, the Drazin inverse and the Moore-Penrose inverse. The inverse along an element is also called Mary inverse along an element. Beside by author and his co-authors, different properties, characterizations and extensions of the inverse along an element was investigated by many others mathematicians in various settings such as semigroups, rings, C*-algebras, Banach algebras, operators, matrices, tensors. The inverse along an element was used to study certain concepts usually based on some specific generalized inverses as a general notation of quotient ring based on inverses along an element.

The definition of the inverse along an element was given in Part II as well as its properties and relations with others generalized inverses. The most of the properties could be

expressed algebraically by using the product operation only and Green's relation. In this part, inverses along centralizers and idempotents (commuting or bicommuting) are studied, with a particular emphasis on idempotents. A proper extension of Cline's formula for such inverses is established. The (*b,c*)-inverse presents a generalization of the inverse along an element. The relation between the (*b,c*)-inverse and the inverse along an element is stated. An extension of Miller and Clifford's theorem is given and its applications are provided to inverse along an element and the (*b,c*)-inverse. Reverse order laws, i.e. finding the inverse of product by using the product of inverses along elements, are investigated in this part too. The author proposes partial orders based on the inverse along an element and the inverse of matrices along triangular matrices. The natural inverse is defined in a ring case to generalize the group and Drazin inverses. Using the additive operation, it is proved that the natural inverse has the unique natural core decomposition which is a generalization of the core-nilpotent decomposition. Also, Jacobson's lemma and absorption law for inverse along an element are developed.

Part III contains the different results of author related to the group inverse as a particular generalized inverse. Various necessary and sufficient conditions for the existence of the group inverse in a semigroup are firstly given in this part. Then, the author presented equivalent conditions for the existence of the group inverse of a product of regular elements in rings, under a simple extra assumption of regularity only. In this case, expressions for the group inverse of such product are established. Furthermore, the problem of the reverse order law for the group inverse of the product *ab* is studied when *a* and *b* are group invertible. Precisely, the problem is when does the formula $(ab)^{\#} = b^{\#}a^{\#}$ hold. This problem is completely solved in stable semigroups and Dedekind-fnite rings. In the general case, the author solves the two-sided reverse order law: $(ab)^{\#} = b^{\#}a^{\#}$ and $(ba)^{\#}=a^{\#}b^{\#}$. One can observe that author relates the reverse order law for the group inverse to Green's pre-orders and Peirce matrix rings. A generalization of unit-regularity by using group invertible elements of the semigroup instead of units (that may even not exist) is exposed too. The connections between special clean elements and group invertible elements of ring are provided. It is proved that the special clean decompositions are in bijective correspondence with completely regular reflexive inverses.

The author has made a great contribution to the theory of generalized inverses. His ideas and proving techniques are authentic and his results are of very high quality and enrich the knowledge in literature. The author's original works inspired many researches and became central topics for a number of scholars. Thus, the author strongly meets the requirements for the Habilitation a Diriger les Recherches



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