Generalized inverses modulo $\mathcal{H}$ in semigroups and rings

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Abstract

The definition of the inverse along an element was very recently introduced, and it contains known generalized inverses such as the group, Drazin and Moore-Penrose inverses. In this paper, we first prove a simple existence criterion for this inverse in a semigroup, and then relate the existence of such an inverse in a ring to the ring units.

Keywords generalized inverses; Green's relations; semigroups; rings
2010 MSC: 15A09; 16E50; 20M99

1 Introduction

In this paper, S is a semigroup and R is a ring with identity. All the definitions are given for the semigroup S and are then used for the ring R, where the ring multiplication acts as the semigroup law. The set of invertible elements of R (simply called "units" in the sequel) will be denoted by R^{-1}. As usual, for a semigroup S, S^1 denotes the monoid generated by S ($R^1 = R$).

We say a is (von Neumann) regular in S if $a \in aSa$. A particular solution to $axa = a$ is called an inner inverse and denoted by a^-. A solution to $xax = a$ is called an outer inverse. Finally, an inner inverse that is also an outer inverse is called reflexive. The set of all inner (resp. outer, resp. reflexive) inverses of a is denoted by $a\{1\}$ (resp. $a\{2\}$, resp. $a\{1, 2\}$).

In [12] a special outer inverse, called inverse along an element, was introduced in the context of semigroups. The purpose of this article is to give new existence criteria of this inverse, notably in the context of rings, where the set of units (invertible elements) comes into play.

We will make use of the Green's preorders and relations in a semigroup [8]. For elements a and b of S, Green's preorders $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{H}}$ are defined by

$$a \leq_{\mathcal{L}} b \Longleftrightarrow S^1 a \subset S^1 b \Longleftrightarrow \exists x \in S^1,\ a = xb;$$
$$a \leq_{\mathcal{R}} b \Longleftrightarrow aS^1 \subset bS^1 \Longleftrightarrow \exists x \in S^1,\ a = bx;$$
$$a \leq_{\mathcal{H}} b \Longleftrightarrow \{a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b\}.$$

If $\leq_{\mathcal{J}}$ is one of these preorders, then $a\mathcal{J}b \Leftrightarrow \{a \leq_{\mathcal{J}} b \text{ and } b \leq_{\mathcal{J}} a\}$, and $\mathcal{J}_a = \{b \in S,\ b\mathcal{J}a\}$ denotes the $\mathcal{J}$-class of a.

We will use the following classical lemmas. Let $a, b, c \in S$.

Lemma 1.1.

$$a \leq_{\mathcal{L}} b \Rightarrow \{\forall x, y \in S^1,\ bx = by \Rightarrow ax = ay\};$$
$$a \leq_{\mathcal{R}} b \Rightarrow \{\forall x, y \in S^1,\ xb = yb \Rightarrow xa = ya\}.$$

Lemma 1.2.

$$ca \leq_{\mathcal{L}} a,\ ac \leq_{\mathcal{R}} a,\ aca \leq_{\mathcal{H}} a;$$
$$a \leq_{\mathcal{L}} b \Rightarrow ac \leq_{\mathcal{L}} bc;$$
$$a \leq_{\mathcal{R}} b \Rightarrow ca \leq_{\mathcal{R}} cb.$$

The definitions of group and Moore-Penrose inverse are the standard in the literature (see, for example, [2], [9]):

1. a is group invertible if there is $a^{\#} \in a\{1,2\}$ that commutes with a;
2. a has a Drazin inverse a^D if a positive power of a is group invertible;
3. if $*$ is an involution in S, then a is Moore-Penrose invertible if there is $a^{\dagger} \in a\{1,2\}$ such that $aa^{\dagger}$ and $a^{\dagger}a$ are symmetric with respect to $*$.

We recall the following characterization of group invertibility in terms of Green's relation $\mathcal{H}$ (see [8], [3]): $a^{\#}$ exists if and only if $a\mathcal{H}a^2$ if and only if $\mathcal{H}_a$ is a group.

In this paper, we study invertibility along a fixed element, as defined recently in [12].

Definition 1.3. *Given a, d in S, we say a is invertible along d if there exists b such that $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. If such an element exists then it is unique and is denoted by $a^{\| d}$.*

This notion generalizes the usual concept of unit, as well as some well known generalized inverses. These arise as special cases (see [12]):

1. $a^{\| 1} = a^{-1}$;
2. $a^{\| a} = a^{\#}$;
3. $a^{\| a^*} = a^{\dagger}$;
4. $a^{\| a^m} = a^D$.

2 A new existence criterion

In [12], the existence of an inverse of a along d was related to the existence of certain group inverses.

Theorem 2.1. *Let $a, d \in S$. Then the following are equivalent:*

1. *$a^{\|d}$ exists.*
2. *$d \leq_{\mathcal{R}} da$ and $(da)^{\#}$ exists.*
3. *$d \leq_{\mathcal{L}} ad$ and $(ad)^{\#}$ exists.*

In this case,

$$b = d(ad)^{\sharp} = (da)^{\sharp} d.$$

Note that if $a^{\|d}$ exists then $d = dab = da(da)^{\sharp} d$ and d is regular.

In this section, we prove another existence criterion, and study its implications.

Theorem 2.2. *Let $a, d \in S$. Then a is invertible along d if and only if $dad\mathcal{H}d$.*

Proof.

- $\Rightarrow$ Suppose a is invertible along d with inverse b. b satisfies $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. By lemma 1.2, $dad \leq_{\mathcal{H}} d$, and we have only to prove the converse inequality. By right compatibilty of the $\leq_{\mathcal{L}}$ preorder (lemma 1.2), $b \leq_{\mathcal{L}} d \Rightarrow bad \leq_{\mathcal{L}} dad$, that is $d \leq_{\mathcal{L}} dad$. Symmetrically, $d \leq_{\mathcal{R}} dad$ and finally $d \leq_{\mathcal{H}} dad$.
- $\Leftarrow$ Suppose $dad\mathcal{H}d$. Then there exist $x, y \in S^1$ such that $d = dadx = ydad$. Let $b = ydadx$. Then $b = dx = yd$ and $b \leq_{\mathcal{H}} d$. Straightforward computations give $bad = ydad = d$, $dab = dadx = d$, and b is the inverse of a along d.

□

It follows that invertibility along an element can be interpreted as a kind of invertibility modulo $\mathcal{H}$:

Definition 2.3. *Let $a, d \in S$. We say that a is an inner inverse of d modulo $\mathcal{H}$, and d is an outer inverse of a modulo $\mathcal{H}$, if $dad\mathcal{H}d$.*

Theorem 2.2 claims that a is an inner inverse of d modulo $\mathcal{H}$ if and only if a is invertible along d. We will denote $d\{1\}[\mathcal{H}] = \{a \in S,\ dad\mathcal{H}d\}$ the set of elements invertible along d. It is important to note that $\mathcal{H}$ is not a congruence in general, and that this notion of invertiblity modulo $\mathcal{H}$ must be taken with care (it is not an equality of $\mathcal{H}$-classes in general). Nevertheless, invertibility modulo $\mathcal{H}$ depends only on the $\mathcal{H}$-class of d, and we can still deduce from the previous characterization an equality regarding $\mathcal{H}$-classes.

Corollary 2.4. *Let $a, c, d \in S$, $c\mathcal{H}d$. Then $dad\mathcal{H}d \Leftrightarrow cac\mathcal{H}c$ and in this case $a^{\|d} = a^{\|c}$.*

Proof. Assume $dad\mathcal{H}d$. Then $a^{\|d}$ exists. By cancellation properties (lemma 1.1), $a^{\|d}ad = d = daa^{\|d}$ implies $a^{\|d}ac = c = caa^{\|d}$, and a is invertible along c with inverse $a^{\|d}$. □

Corollary 2.5.

$$dad\mathcal{H}d \Leftrightarrow \mathcal{H}_d a \mathcal{H}_d = \mathcal{H}_d$$

Proof. Let $a \in d\{1\}[\mathcal{H}]$ and $c, c' \in \mathcal{H}_d$. By corollary 2.4 a is invertible along c and along c', $cac\mathcal{H}c$ and $c'ac'\mathcal{H}c'$. We have to show that $cac' \in \mathcal{H}_d$. But by lemma 1.2

$$c \leq_{\mathcal{R}} c' \Rightarrow cac \leq_{\mathcal{R}} cac' \Rightarrow c \leq_{\mathcal{R}} cac';$$
$$c' \leq_{\mathcal{L}} c \Rightarrow c'ac' \leq_{\mathcal{L}} cac' \Rightarrow c' \leq_{\mathcal{L}} cac'.$$

But $c\mathcal{H}c'\mathcal{H}d$ hence $d \leq_{\mathcal{H}} cac' \leq_{\mathcal{H}} d$.
The converse implication is straightforward. □

Applying theorem 2.2 to the classical generalized inverses we get:

Corollary 2.6. *$a^{\#}$ exists if and only if $a^3\mathcal{H}a$, a^D exists if and only if there exists a positive integer m such that $a^{2m+1}\mathcal{H}a^m$, and $a^{\dagger}$ exists if and only if $a^*aa^*\mathcal{H}a^*$.*

Proof. We simply use the characterizations of the classical generalized inverses in terms of inverses along an element: $a^{\#} = a^{\|a}$, $a^D = a^{\|a^m}$ for some positive integer m and $a^{\dagger} = a^{\|a^*}$. □

Note that the criterion of existence for the group inverse is classical, since the equivalence $a^3\mathcal{H}a \Leftrightarrow a^2\mathcal{H}a$ is straightforward. For the Drazin inverse, it is also direct to prove that $a^{2m+1}\mathcal{H}a^m \Leftrightarrow a^{m+1}\mathcal{H}a^m$. An element satisfying this last equation is said to satisfy Azumaya's property of strongly π-regularity [1]. The link with Drazin invertibilty was made by Drazin in its seminal paper [4]. The condition for the Moore-Penrose inverse can be derived directly from a result of Puystjens and Robinson [16]: $a^{\dagger}$ exists if and only if $a \in aa^*R \cap Ra^*a$.

3 Creation of units in a ring

In the ring case, we take advantage of the ring structure to characterize the existence of $a^{\|d}$ by means of a unit in the ring R and elements of $d\{1\}$ (inner inverses of d). Recall that $a^{\|d}$ exists implies d is regular.

We will make use of the well known Jacobson's lemma (see for instance [11]) that traces back to his work on the radical of a ring [10]:

Lemma 3.1. *$1 - xy$ is a unit iff $1 - yx$ is a unit, in which case $(1 - xy)^{-1} = 1 + x(1 - yx)^{-1}y$.*

Theorem 3.2. *Let d be a regular element of a ring R, $d^- \in d\{1\}$. Then the following are equivalent:*

1. *$a^{\|d}$ exists.*
2. *$u = da + 1 - dd^-$ is a unit.*
3. *$v = ad + 1 - d^-d$ is a unit.*

In this case,

$$a^{\|d} = u^{-1}d = dv^{-1}.$$

Proof.

(2) $\Leftrightarrow$ (3) This is Jacobson's lemma. Pose $x = -d$ and $y = a - d^-$. Then

$$u = 1 + da - dd^- = 1 - xy \text{ is a unit } \Leftrightarrow 1 - yx = 1 + ad - d^-d = v \text{ is a unit.}$$

(1) $\Rightarrow$ (2) If $a^{\|d}$ exists then $dad\mathcal{H}d$, and there exist $x, y \in R$ such that $d = dadx = ydad$. Since $(dadd^- + 1 - dd^-)(dxd^- + 1 - dd^-) = 1 = (ydd^- + 1 - dd^-)(dadd^- + 1 - dd^-)$ then $dadd^- + 1 - dd^-$ is a ring unit. But $dadd^- + 1 - dd^- = 1 - (1 - da)dd^-$ and by Jacobson lemma 3.1, $u = dd^-da + 1 - dd^- = 1 + dd^-(1 - da)$ is a unit.

(2) $\Rightarrow$ (1) We know (2) and (3) are equivalent, so assume u and v are units. Computations give $ud = dad = dv$, hence $d = u^{-1}dad = dadv^{-1}$. This means exactly that $dad\mathcal{H}d$ that is (theorem 2.2) $a^{\|d}$ exists.

Finally, $ud = dad = dv \Rightarrow u^{-1}d = dv^{-1}$. Let $b = u^{-1}d = dv^{-1}$. Then $b \leq_{\mathcal{H}} d$ and $bad = u^{-1}dad = d = dadv^{-1} = dab$, that is $b = u^{-1}d = dv^{-1} = a^{\|d}$. □

Corollary 3.3. *Given a regular element $a \in R$, and $a^- \in a\{1\}$,*

1. *$a^{\#} = a^{\|a}$ exists if and only if $a^2 + 1 - aa^-$ is a unit, or equivalently, $a^2 + 1 - a^-a$ is a unit;*
2. *$a^{\dagger} = a^{\|a^*}$ exists if and only if $aa^* + 1 - a^*(a^*)^-$ is a unit, or equivalently, $a^*a + 1 - (a^*)^-a^*$ is a unit.*

These are not the classical existence criteria for the group [15] and Moore-Penrose inverses [13]. However, we can recover the classical existence criterion for the Moore-Penrose inverse as follows: pick $(a^*)^- = (a^-)^*$ as an inner inverse of a^*, and just tranpose $u = a^*a + 1 - (a^*)^-a^*$. We get $u^* = a^*a + 1 - aa^-$ is a unit, which is the classical relation [13].

For the group inverse, we study the invertibilty of 1 along a rather than the invertibility of a along a to recover the classical relation [15].

Corollary 3.4. *The following statements are equivalent:*

1. *$a^{\sharp}$ exists (a is group invertible);*
2. *$1^{\|a}$ exists;*

3. $a + 1 - aa^-$ *is a unit for any* $a^- \in a\{1\}$.

Proof. We already know that (2) $\Leftrightarrow$ (3). But the equivalence of (1) and (2) follows from the characterization of group invertibility in terms of relation $\mathcal{H}$, and the existence criterion of $1^{\|a}$ of theorem 2.2: $a^\sharp\exists \iff a^2\mathcal{H}a \iff a1a\mathcal{H}a \iff 1^{\|a}\exists$. □

4 Application: Unit regular elements

An interesting application of the previous results concerns unit regular elements. Recall that $d \in R$ is unit regular if $d \in dR^{-1}d$. Unit-regular elements and unit-regular rings first appeared in [7]. Unit regular elements were also studied by Harte ([5], [6]) under the name "decomposably regular" elements. We now prove that unit regularity modulo $\mathcal{H}$ is sufficient to be unit regular, and derive existence criteria from theorem 3.2.

Theorem 4.1. *Let* d *be a regular element of a ring* R, $d^- \in d\{1\}$. *Then the following are equivalent:*

1. $d \in dR^{-1}d$.
2. $\exists a \in R^{-1},\ dad\mathcal{H}d$.
3. $\exists a \in R^{-1},\ u = da + 1 - dd^-$ *is a unit.*
4. $\exists a \in R^{-1},\ v = ad + 1 - d^-d$ *is a unit.*

Proof. We already know that (2), (3) and (4) are equivalent. (1) implies (2) is trivial. Let us prove that (2) implies (1).
Let $a \in R^{-1}$, $dad\mathcal{H}d$. This means that a is invertible along d, d is regular and by theorem 3.2, for any $d^- \in d\{1\}$, $u = da + 1 - dd^-$ is a unit and $u^{-1}d = a^{\|d}$. From $daa^{\|d} = d$, we get $dau^{-1}d = d$ and d is unit regular, with inverse $x = au^{-1} \in R^{-1}$. □

For instance, group invertibles elements, being unit regular modulo $\mathcal{H}$ ($d1d\mathcal{H}d \Leftrightarrow d$ is group invertible), are unit regular (see [6] for an alternative proof).

Finally, we remark that in the case of a Banach algebra A, this theorem shows that we can take $\mathrm{cl}(A^{-1})$ (closure of the group of units) instead of A^{-1} in the definition of unit regular elements.

Corollary 4.2. $d \in dA^{-1}d \iff d \in d\mathrm{cl}(A^{-1})d$.

Proof. The implication is straightforward. For the converse, assume that d has a inner inverse in the closure of A^{-1}, $d^- \in d\{1\} \cap \mathrm{cl}(A^{-1})$. Then exists $a \in A^{-1}$, $||a - d^-||.||d|| < 1$ and $v = 1 + (a - d^-)d$ is invertible. By theorem 4.1, d is unit regular. □

Aknowlegdments

This research was financed by FEDER Funds through "Programa Operacional Factores de Competitividade – COMPETE" and by Portuguese Funds through FCT - "Fundação para a Ciência e a Tecnologia", within the project PEst-C/MAT/UI0013/2011.
The authors thank the referee and the editor for providing a shorter proof of theorem 3.2.

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