Theory of Subdualities

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Abstract

We present a new theory of a dual systems of vector spaces that extends the existing notions of reproducing kernel Hilbert spaces and Hilbert subspaces. In this theory kernels (understood as operators rather than kernel functions) need not to be positive nor self-adjoint. These dual systems called subdualities hold many properties similar to those of Hilbert subspaces and treat the notions of Hilbert subspaces or Kreǐn subspaces as particular cases. Some applications to Green operators or invariant subspaces are given.

Keywords Subdualities, Hilbert subspaces, reproducing kernels, duality

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Introduction

Functions of two variables appearing in integral transforms (Zaremba, Bergman, Segal, Carleman), or more generally kernels in the sense of Laurent Schwartz [37] - defined as weakly continuous linear mappings between the dual of a locally convex vector space and itself - have been investigated for nearly a century and have interplay with many branches of mathematics: distribution theory [37], differential equations [13], probability theory [39], [24], [29], approximation theory [23], [17] but also harmonic analysis and Lie theory [38], [15], operator theory [3], [35] or geometric modeling [28], [31].

The study of these objects may take various forms, but in case of positive kernels, the study of the properties of the image space initiated by Moore, Bergman and Aronszajn[7] leads to a crucial result: the range of the kernel can

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be endowed with a natural scalar product that makes it a prehilbertian space and its completion belongs (under some weak additional conditions either on the kernel or on the locally convex space) to the locally convex space. Moreover, this injection is continuous. Positive kernels then seem to be deeply related to some particular Hilbert spaces and our aim in this article is to study the other kernels. What can we say if the kernel is neither positive, nor Hermitian ?

To do this we study directly spaces rather than kernels. Considering Hilbert spaces, some mathematicians - among them Aronszajn [6], [7] and Schwartz [40], [37] - have been interested in a particular subset of the set of Hilbert spaces, those Hilbert spaces that are continuously included in a common locally convex vector space. The relative theory is known as the theory of Hilbert subspaces and its main result is that surprisingly the notions of Hilbert subspaces and positive kernels are equivalent under the (weak) hypothesis of quasi-completeness of the locally convex space, which is generally summarized as follows: "there exists a bijective correspondence between positive kernels and Hilbert subspaces".

Moreover this theory has been generalized to the Hermitian case by Laurent Schwartz [37] this leading to a most more complicated theory of Hermitian subspaces. This spaces are also called nowadays Kreĭn subspaces [1] (or Pontryagin subspaces in the finite-dimensional case [41]) for their link with Kreĭn spaces, see [9] or [8].

In this article we present a new theory of a dual system of vector spaces called subdualities (see [26] for a first introduction) which deals with the notion of Hilbert or Hermitian subspaces as particular cases. A topological definition (Proposition 1.3) of subdualities is as follows: a duality (E, F) is a subduality of the dual system $(\mathcal{E}, \mathcal{F})$ if both E and F are weakly continuously embedded in $\mathcal{E}$. It appears that we can associate a unique kernel (in the sense of L. Schwartz, Theorem 1.11) with any subduality, whose image is dense in the subduality (Theorem 1.17). The study of the image of a subduality by a weakly continuous linear operator (Theorem 2.2), makes it possible to define a vector space structure upon the set of subdualities (Theorem 2.3), but given a certain equivalence relation. A canonical representative entirely defined by the kernel is then given (Theorem 3.4), which enables us to state a bijection theorem between canonical subdualities and kernels.
We also study the particular case of subdualities of $\mathbb{K}^{\Omega}$ which we name evaluation (or reproducing kernel) dualities. Their kernel may then be identified with a kernel function (definition 1.13).
Such subdualities and kernels appear for instance in the study of polynomial spaces, Chebyshev splines and blossoming, see for instance M-L. Mazure and P-J. Laurent [28].

Finally we connect this theory we some more or less recent works on Hilbert subspaces and study normal subdualities (see [37] for normal Hilbert subspaces) and the relative concept of Green operator,but also representation theory and invariant subdualities. This field, together with the one of Kreĭn subspaces are very active (see for instance [4], [5] for Kreĭn subspaces and [15], [32], [42], [44] for invariant Hilbert subspaces and representation theory).

This work brings up many questions, with both theoretical and applied insights. Many questions are devoted to canonical subdualities: is there an easy characterization of canonical subdualities, are they interesting enough, can one characterize directly stable kernels ? Other questions deal with differential operators and their link with Sobolev spaces, or group representations. The concept of Green operator associated to a kernel or the Berezin symbol of operators in evaluation dualities are also of interest.

Conventions and notations

The theory of Hilbert subspaces and more generally the theory of subdualities, as its name indicates, relies mainly on the duality theory for topological vector spaces. Therefore we will only consider locally convex (Hausdorff) topological vector spaces or (Hausdorff) dualities. Throughout this study $\mathcal{E}$ will always be a locally convex (Hausdorff) topological vector space (in short l.c.s.) over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $(\mathcal{E}, \mathcal{F})$ a dual system of vector spaces.

In order to be able to deal with inner product spaces, hence sesquilinear forms, any complex vector space $\mathcal{E}$ (i.e. over the scalar field $\mathbb{C}$) will be endowed with a continuous anti-involution (conjugation) $C_j : \mathcal{E} \longrightarrow \overline{\mathcal{E}}$ when needed such that $\overline{\mathcal{E}} = \mathcal{E}$. This will however not be the case in general.

We have here chosen to deal with bilinear forms and kernels are linear mappings between the dual of a l.c.s. and itself, or between the two spaces defining a given duality. An other completely acceptable choice would have been to treat sesquilinear forms and semi-dualities. Kernels would then be linear mappings between the anti-dual of a l.c.s. and itself, or between the two spaces defining a given semi-duality. This is the point of view taken for the study of Hilbert subspaces, see [37].

1 Subdualities and associated kernels

In this section, we introduce a new mathematical object that we call subduality of a dual system of vector spaces (or equivalently subduality of a locally convex topological vector space). These objects appear to be closely linked with kernels (Theorem 1.11 and Lemma 1.16) and could therefore be the appropriate setting to study such linear applications.

1.1 Subdualities of a dual system of vector space

The definition of subdualities remains heavily on the definition of a duality that therefore is restated below.

Definition 1.1 *Two vector spaces E, F are said to be in duality if there exists a bilinear form L on the product space $F \times E$ separate in E and F, i.e.:*

1. $\forall e \neq 0 \in E, \exists f \in F,\ L(f,e) \neq 0$;
2. $\forall f \neq 0 \in F, \exists e \in E,\ L(f,e) \neq 0$.

In this case, (E,F) is said to be a duality (relative to L).

The following morphisms are then well defined:

$$\begin{array}{rcl} \gamma_{(E,F)} : F & \longrightarrow & E^* \text{ algebraic dual of E} \\ y & \longmapsto & L(y,.) \end{array} \qquad \begin{array}{rcl} \theta_{(E,F)} : E' \triangleq \gamma_{(E,F)}(F) & \longrightarrow & F \\ L(y,.) & \longmapsto & y \end{array}$$

We can now give the definition of subdualities. Subdualities may be seen as completely algebraic objects and therefore the first definition is purely algebraic. $\forall A \subset \mathcal{E},\ u_{|A}$ denotes the restriction of u to the set A.

Definition 1.2 (– subduality –)
Let (E,F) and $(\mathcal{E},\mathcal{F})$ be two dualities.
(E,F) is a subduality of $(\mathcal{E},\mathcal{F})$ if:

- $E \subseteq \mathcal{E}$; $\qquad F \subseteq \mathcal{E}$;
- $\gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|E}) \subseteq \gamma_{(E,F)}(F)$; $\qquad \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|F}) \subseteq \gamma_{(F,E)}(E)$.

We note $\mathcal{SD}((\mathcal{E},\mathcal{F}))$ the set of subdualities of $(\mathcal{E},\mathcal{F})$.

The two first conditions are simply that E and F, as vector spaces, are algebraically included in the reference vector space $\mathcal{E}$.

The last conditions deal with inclusions for the linear forms: they state that every vector of $\mathcal{F}$, as a linear form on $E \subset \mathcal{E}$ (resp. on $F \subset \mathcal{E}$), is in F (respectively in E), *i.e*

$$\forall \varphi \in \mathcal{F},\ \exists f \in F,\ \forall e \in E,\ (\varphi, e)_{(\mathcal{F},\mathcal{E})} = (f,e)_{(F,E)}$$

Remark that such an f is unique by the Hausdorff property.

If $\mathcal{E}$ is a locally convex space, we say that (E,F) is a subduality of $\mathcal{E}$ if it is a subduality of $(\mathcal{E},\mathcal{E}')$ and we denote by $\mathcal{SD}(\mathcal{E})$ the set of subdualities of the l.c.s. $\mathcal{E}$.

We will sometimes use the following notations $(E,F) \hookrightarrow (\mathcal{E},\mathcal{F})$ (resp. $(E,F) \hookrightarrow \mathcal{E}$) to say that (E,F) is a subduality of the dual system $(\mathcal{E},\mathcal{F})$ (resp. of the l.c.s. $\mathcal{E}$).

We can also interpret the previous algebraic inclusions in topological terms, since dualities make a bridge between topological and algebraic properties. An equivalent topological definition of subdualities is then included in the following theorem:

Theorem 1.3 *The following statements are equivalent:*

1. (E,F) *is a subduality of* $(\mathcal{E},\mathcal{F})$,
2. *The canonical injections* $i : E \mapsto \mathcal{E}$ *and* $j : F \mapsto \mathcal{E}$ *are weakly continuous,*
3. $i : E \mapsto \mathcal{E}$ *et* $j : F \mapsto \mathcal{E}$ *are continuous with respect to the Mackey topologies on* E, F *and* $\mathcal{E}$.

The equivalence between (1) and (3) is notably useful in case of metric spaces, since any locally convex metrizable topology is the Mackey topology (Corollary p 149 [20] or Proposition 6 p 71 [10]). In case of subdualities of a locally convex space, one must notice that the initial topology plays no role in the definition, that emphases the role of the dual system $(\mathcal{E},\mathcal{E}')$ only.

Proof Let us show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1):

(1) $\Rightarrow$ (2) We define the following mappings (canonical inclusions):

$$i : E \mapsto \mathcal{E}, \qquad j : F \mapsto \mathcal{E},$$
$$i' : \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}) \mapsto \gamma_{(E,F)}(F), \qquad j' : \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}) \mapsto \gamma_{(F,E)}(E)$$

i and i' (resp. j and j') are transposes for the weak topology hence weakly continuous since $\forall \varepsilon' \in \mathcal{E}' = \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F})$, $\exists i'(\varepsilon') \in E' = \gamma_{(E,F)}(F)$, $\forall e \in E$:

$$(\varepsilon', i(e))_{\mathcal{E}',\mathcal{E}} = (i'(\varepsilon'), e)_{E',E}$$

that is exactly the definition of the transpose. It is the classical link between inclusion of the topological dual and weak continuity.

(2) ⇒ (3) Since i' (resp.j') is weakly continuous, its transpose is continuous for the Mackey topologies (Corollary 3 p 111 [20]). We could also cite Corollary 2 p 111 [20]: if $u : E \mapsto \mathcal{E}$ is weakly continuous, then it is continuous if E is endowed with the Mackey topology and $\mathcal{E}$ with any compatible topology).

(3) ⇒ (1) Since $i : E \mapsto \mathcal{E}$ and $j : F \mapsto \mathcal{E}$ are continuous for the Mackey topologies, their transposes ${}^t i : \mathcal{E}' \mapsto E'$ and ${}^t j : \mathcal{E}' \mapsto F'$ exist. But $\mathcal{E}' = \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F})$ and $E' = \gamma_{(E,F)}(F)$ (resp. $F' = \gamma_{(F,E)}(E)$) since the Mackey topology is compatible with the duality, that prove the result.

Remark 1.4 *If (E, F) is a subduality of $(\mathcal{E}, \mathcal{F})$, then (F, E) is also a subduality of $(\mathcal{E}, \mathcal{F})$.*

Of special interest are the subdualities of genuine functions where the evaluation functionals $\delta_t : f \mapsto f(t)$ are continuous. We call them evaluation dualities; They will later also be called reproducing kernel duality due to a forthcoming property.

Definition 1.5 (– evaluation duality –)
Let Ω be any set. We call evaluation duality on Ω any subduality of $\mathbb{K}^\Omega$ endowed with the product topology (topology of simple convergence).

example 1 **Polynomials, splines**
In [31] the authors consider the spaces $E_\mathcal{P} = F_\mathcal{P} = \mathcal{P}_n$ of real polynomials of degree n and the following bilinear form on $F_\mathcal{P} \times E_\mathcal{P}$

$$\begin{array}{rcl} L : F_\mathcal{P} \times E_\mathcal{P} & \longrightarrow & \mathbb{R} \\ (f, e) & \longmapsto & \displaystyle\sum_{j=0}^{n} \frac{(-1)^{n-j}}{n!} f^{(j)}(\tau) e^{(n-j)}(\tau) \end{array}$$

that does not depend on the particular point τ chosen.
It is straightforward to see that this duality is separate (by using the monomials) and that $E_\mathcal{P}$ and $F_\mathcal{P}$ endowed with the weak-topology are continuously included in the l.c.s. $\mathbb{R}^\mathbb{R}$ endowed with the topology product. $(E_\mathcal{P}, F_\mathcal{P})$ is then a subduality of $\mathbb{R}^\mathbb{R}$, *i.e.* an evaluation duality on $\mathbb{R}$.

example 2 **Entire functions and Hermite polynomials**
Let H_n denote the Hermite polynomials on $\mathbb{C}$, and define

$$E_H = \{e = \sum_{n\in\mathbb{N}} \alpha_n \frac{H_n}{n!},\ \{|\alpha_n|^{\frac{1}{n}}, \mathrm{n} \in \mathbb{N}\} \in l^\infty(\mathbb{C})\}$$

Let also F_H be the vector space of entire functions,

$$F_H = \{f(z) = \sum_{n\in\mathbb{N}} \beta_n z^n,\ \sum_{n\in\mathbb{N}} |\beta_n| z^n < +\infty\, \forall z \in \mathbb{C}\}$$

These two vector spaces may be put in duality by the following bilinear form

$$\begin{array}{rcl} L : F_H \times E_H & \longrightarrow & \mathbb{C} \\ (f, e) & \longmapsto & \sum_{n\in\mathbb{N}} \alpha_n \beta_n \end{array}$$

since this sum is absolutely convergent. A representative for the evaluation functional $\delta_w : e \in E_H \longmapsto e(w)$ is given by $\phi(z) = \sum_{n\in\mathbb{N}} \frac{H_n(w)z^n}{n!}$, $\phi \in F_H$, whereas a representative for the evaluation functional $\delta_z : f \in F_H \longmapsto f(z)$ is given by $\psi(w) = \sum_{n\in\mathbb{N}} \frac{H_n(w)z^n}{n!}$, $\psi \in E_H$.

It follows that (E_H, F_H) is an evaluation duality over $\mathbb{C}$. We will give an interpretation of the two-variable function in section 1.4. This bilinear form has a interpretation in terms of Malliavin calculus [25].

example 3 **Harmonic and Hyperharmonic functions**
This example is based on the article [21].
Let $m, n \in \mathbb{N}$, $m > n - 1$ and define the measure $d\nu_m$ on the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ by

$$d\nu_m(x) = \frac{2(1-|x|^2)^{m-n}}{n\beta(\frac{n}{2}, m+1-n)} d\nu(x)$$

where $d\nu$ is the normalized Lebesgue measure on B and $\beta(.,.)$ the Euler beta function.

Let

$$H(B) = \{u \in C^2(B),\ \Delta(u) = 0\}$$

and

$$h(B) = \{u \in C^2(B),\ \Delta_h(u) = 0\}$$

be the sets of harmonic and hyperharmonic functions on B.

In [21] the authors proved the existence of a two-variable function kernel function $K_m(x, y)$ on B verifying:

$$\forall f \in H(B) \bigcap L^1(B, d\nu_m), \quad f(y) = \int_B K_m(x, y) f(x) d\nu_m(x), \quad y \in B$$

$$\forall g \in h(B) \bigcap L^1(B, d\nu_m), \quad g(x) = \int_B K_m(x, y) g(y) d\nu_m(y), \quad x \in B$$

and gave an expression of K in terms of extended zonal harmonics. Going further in the study of the kernel, we can show that:

$$\forall x \in B,\ K_m(x, .) \in H(B) \bigcap L^\infty(B)$$
$$\forall y \in B,\ K_m(., y) \in h(B) \bigcap L^1(B)$$

This proves that the bilinear form

$$(g, f) = \int_B g(x)f(x)d\nu_m(x)$$

is well defined and separate on $h(B) \bigcap L^1(B) \times H(B) \bigcap L^\infty(B)$, and that the evaluation functionals are weakly continuous.

Putting all the results together, we have:

Theorem 1.6 *The duality* $(E_m = H(B) \bigcap L^\infty(B), F_m = h(B) \bigcap L^1(B))$ *with bilinear form* $(g, f)_{(F_m, E_m)} = \int_B g(x)f(x)d\nu_m(x)$ *is a subduality of* $\mathbb{R}^B$ *(endowed with the product topology), i.e. an evaluation duality on* B*.*

Once again we will see that the two-variable function $K_m(x, y)$ plays a great role in section 1.4 and explain its name as kernel function.

1.2 Inner product spaces

In this section any space $\mathcal{E}$ will be endowed with a continuous anti-involution.

An other class of important examples is given by inner product spaces. Recall that an inner product space H is a vector space endowed with a non-degenerate Hermitian sesquilinear form. This inner product puts H in duality with its conjugate space $\overline{H}$ with respect to the bilinear form on $\overline{H} \times H$:

$$\left(\overline{h_1}, h_2\right)_{(\overline{H}, H)} = L(\overline{h_1}, h_2) = \langle h_1 | h_2 \rangle_H$$

In case the inner product is positive, one must be careful that the norm-topology defined by the inner product is not compatible with the duality in case H is not complete for this norm.

The classical theory deals with Hilbert subspaces, whose definition is restated below:

Definition 1.7 *Let* $(\mathcal{E}, \mathcal{F})$ *be a duality. Then* H *is a Hilbert subspace of* $(\mathcal{E}, \mathcal{F})$ *if and only if* H *is an algebraic vector subspace of* $\mathcal{E}$ *endowed with a definite positive inner product that makes it a Hilbert space and such that the canonical injection is weakly continuous.*
Reproducing kernel Hilbert spaces are the Hilbert subspaces of $\mathbb{K}^\Omega$*.*

But we can find also in the literature the notion of Hermitian subspace [37] or equivalently Kreǐn subspace [41],[1] where the inner product is indefinite. Recall that a Kreĭn space may be seen as the direct difference of two Hilbert spaces. When the dimension of the negative space is finite, it is also called a Pontryagin space.

Definition 1.8 *Let* $(\mathcal{E}, \mathcal{F})$ *be a duality. Then* H *is a Hermitian subspace of* $(\mathcal{E}, \mathcal{F})$ *if and only if* H *is an algebraic vector subspace of* $\mathcal{E}$ *endowed with an indefinite positive inner product that makes it a Kreĭn space and such that the canonical injection is weakly continuous.*

Now let H be an inner product space in duality with its conjugate space. If H is weakly continuously included in $\mathcal{E}$, then so is $\overline{H}$ thanks to the existence of a continuous anti-involution hence any Hilbert subspace or Kreĭn subspace H of $\mathcal{E}$ defines a subduality $(H, \overline{H})$. If moreover $H = \overline{H}$, H may also be put in (only conjugate symmetric) duality with itself and define a "self-subduality" (H, H). **The concepts of Hilbert subspaces, Kreĭn subspaces or prehermitian subspaces are then particular cases of the more general notion of subdualities:**

Theorem 1.9 *Let* H *be an inner product space,* $(H, \overline{H})$ *the duality induced by the inner product. Then* $(H, \overline{H})$ *is a subduality of the dual system* $(\mathcal{E}, \mathcal{F})$ *if and only if* H *is weakly continuously included in* $\mathcal{E}$*. In this case, we say that the inner product space* H *is a self-conjugate subduality of* $(\mathcal{E}, \mathcal{F})$*.*

Proof Evident since $E = H = \overline{F}$ and the bilinear form is conjugate symmetric.

1.3 Kernels

Subdualities are highly linked with kernels, understood as weakly continuous mappings between the two spaces forming a duality, or equivalently between the dual space of a l.c.s. and itself. This section restates the basic definitions and results concerning kernels.

Definition 1.10 (− kernel −)
We call <u>*kernel*</u> *relative to a duality* $(\mathcal{E}, \mathcal{F})$ *(and note* $\varkappa : \mathcal{F} \longrightarrow \mathcal{E}$*) any weakly continuous linear application from* $\mathcal{F}$ *into* $\mathcal{E}$*.*

The definition of a kernel relative to a locally convex space follows, since any l.c.s. $\mathcal{E}$ defines a duality $(\mathcal{E}, \mathcal{E}')$.

Since a kernel is weakly continuous, it has a transpose ${}^t\varkappa$ and an adjoint $\varkappa^* = \overline{{}^t\varkappa}$ when there is an involution. But from the definition of a kernel its transpose and adjoint are also kernels of the duality $(\mathcal{E}, \mathcal{F})$ and we can define the symmetry, self-adjoint and positiveness properties.

The space of kernels of the dual system $(\mathcal{E}, \mathcal{F})$ is denoted by $\mathbf{L}(\mathcal{F}, \mathcal{E})$, $\mathbf{L}(\mathcal{E}', \mathcal{E})$ or simply $\mathbf{L}(\mathcal{E})$, for kernels of the l.c.s. $\mathcal{E}$.

Once again, the space $\mathbb{K}^\Omega$ holds a special place regarding kernels, for they can be identified with kernel functions:

$$\mathbf{L}((\mathbb{K}^\Omega)', \mathbb{K}^\Omega) \cong \mathbb{K}^{\Omega\times\Omega}$$

The wanted isomorphism is given by $[u(\delta_t)](s) = \tilde{u}(t,s)\,\forall t, s \in \Omega$.

example 1 **$\mathbb{R}^n$-example**
Let $(\mathcal{E}, \mathcal{F}) = (\mathbb{R}^n, \mathbb{R}^n)$ in Euclidean duality. Any kernel $\varkappa$ may then be identified with a matrix K of $\mathcal{M}_n(\mathbb{R})$ by $K(i,j) = (e_i, \varkappa(e_j))_{(\mathcal{F},\mathcal{E})}$

example 2 **kernel theorem**
Let $\mathcal{E} = D'(\Omega)$ be the space of distribution on an open set Ω of $\mathbb{R}$. Then we can identify its dual with the set of test functions $\mathcal{F} = D(\Omega) = C_0^\infty(\Omega)$ and by the kernel theorem of L. Schwartz the set of kernels of $D'(\Omega)$ is isomorphic with the set of distributions on $\Omega \times \Omega$:

$$\varkappa : \phi \mapsto \varkappa(\phi)(.) = \int_\Omega K(., s)\phi(s)ds$$

where K is a distribution on $\Omega \times \Omega$. (There exists a general form of this theorem related to tensor products see [19], [43]).

1.4 The kernel of a subduality

A key result concerning Hilbert subspaces is their link with positive kernels. Regarding subdualities, we can also state an important theorem that associates a kernel to each subduality:

Theorem 1.11 (– kernel of a subduality –)
Each subduality (E, F) of $(\mathcal{E}, \mathcal{F})$ is associated with a unique kernel $\varkappa$ of $(\mathcal{E}, \mathcal{F})$ verifying

$$\forall f \in F, \forall \varphi \in \mathcal{F}, \quad (\varphi, j(f))_{(\mathcal{F},\mathcal{E})} = \left(f, i^{-1}\varkappa(\varphi)\right)_{(F,E)}$$

called kernel of the subduality (E, F) of $(\mathcal{E}, \mathcal{F})$. It is the linear application

$$\begin{array}{rcl} \varkappa : \mathcal{F} & \longrightarrow & \mathcal{E} \\ \varphi & \longmapsto & i \circ \theta_{(F,E)} \circ {}^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi) \end{array}$$

considering transposition in the topological dual spaces or simply

$$\begin{array}{rcl} \varkappa : \mathcal{F} & \longrightarrow & \mathcal{E} \\ \varphi & \longmapsto & i \circ^t j(\varphi) \end{array}$$

considering transposition in dual systems.

Proof If we consider transposition in the topological duals:
$\forall f, \in F,\ \varphi \in \mathcal{F}$

$$\begin{aligned} (\varphi, j(f))_{(\mathcal{F},\mathcal{E})} &= ({}^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi), f)_{(F',F)} \\ &= (f, \theta_{(F,E)} \circ^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi))_{(F,E)} \\ &= (f, i^{-1}(i \circ \theta_{(F,E)} \circ^t j) \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi))_{(F,E)} \end{aligned}$$

The solution is unique since $L(.,.) = (.,.)_{(F,E)}$ separates E and F and

$$\varkappa = i \circ \theta_{(F,E)} \circ^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}$$

If we consider transposition in dual systems, then the proof reduces to:

$$(\varphi, j(f))_{(\mathcal{F},\mathcal{E})} = ({}^t j(\varphi), f)_{(E,F)} = \left(f, i^{-1} \circ i \circ^t j(\varphi)\right)_{(F,E)}$$

Finally, $\varkappa$ is weakly continuous by composition of weakly continuous linear applications.

The concept of subduality and of its associated kernel is illustrated by figure 1 and figure 2. In figure 1 we consider transposition in the topological dual spaces and in figure 2 transposition in dual systems.

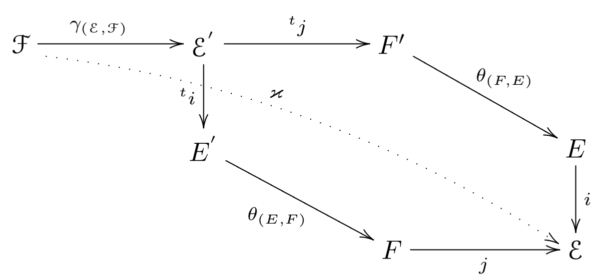

Figure 1: Illustration of a subduality, the relative inclusions and its kernel.

Note that these diagrams are not commutative (the path below is associated to ${}^t\varkappa$) unless the kernel $\varkappa$ is symmetric.

From now on and for the sake of simplicity, we will always consider transposition in dual systems unless explicitly stated.

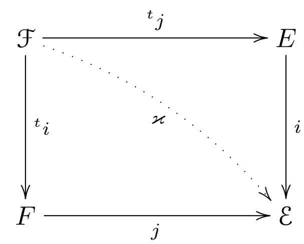

Figure 2: Illustration of a subduality and its kernel (transposition in dual systems).

We can then define the application

$$\begin{array}{rcl} \Phi : \mathcal{SD}((\mathcal{E}, \mathcal{F})) & \longrightarrow & \mathbf{L}(\mathcal{F}, \mathcal{E}) \\ (E, F) & \longmapsto & \varkappa \end{array}$$

that associates to each subduality its kernel. It is a well defined function.

The following lemma can then be deduced directly from theorem 1.3:

Lemma 1.12 *$\varkappa : \mathcal{F} \longrightarrow E$ is weakly continuous if E and $\mathcal{F}$ are equipped with:*

1. *the weak topologies,*
2. *the Mackey topologies.*

We have seen previously that (F, E) is also a subduality of $(\mathcal{E}, \mathcal{F})$. Its kernel is the linear application $\widetilde{\varkappa} = j \circ^t i$ i.e. $\widetilde{\varkappa} = {}^t\varkappa$.

example 1 **Sobolev spaces**
Suppose $\Omega =]0, 1[$. The kernel of the subduality $(E_W, F_W) \hookrightarrow (D'(]0, 1[), D(]0, 1[))$ where

$$E_W = \left\{ e \in D',\ e(s) = \int_\Omega 1\!\!1_{t \leq s} \phi(t) dt, \phi \in L^2(\Omega) \right\}$$

and

$$F_W = \left\{ f \in D',\ f(t) = \int_\Omega 1\!\!1_{t \leq s} \psi(s) ds, \psi \in L^2(\Omega) \right\}$$

are in duality with respect to the bilinear form

$$(f, e)_{(F_W, E_W)} = \int_\Omega \psi(u) \phi(u) du$$

is the integral operator

$$\begin{array}{rcl} \varkappa_W : D(]0,1[) & \longrightarrow & D'(]0,1[) \\ \varphi & \longmapsto & \varkappa_W(\varphi)(.) = \int_\Omega K_W(t,.)\varphi(t)dt \end{array}$$

where $K_W(t,s) = (s-t)1\!\!1_{t\leq s}$.
The kernel of (F_W, E_W) is defined by the distribution

$${}^tK_W(t,s) = (t-s)1\!\!1_{s\leq t} = K_W(s,t)$$

example 2 **The fundamental example of a Hilbert space**
We suppose that $\mathcal{E}$ is endowed with a continuous anti-involution such that $\bar{\mathcal{E}} = \mathcal{E}$. Let H be a Hilbert subspace of $(\mathcal{E}, \mathcal{F})$ and define the following bilinear form on $\overline{H} \times H$ such that $(H, \overline{H})$ is a duality:

$$\begin{array}{rcl} L : \overline{H} \times H & \longrightarrow & \mathbb{K} \\ \overline{h_1}, h_2 & \longmapsto & \langle h_1 | h_2 \rangle \end{array}$$

$(H, \overline{H})$ is a subduality of $(\mathcal{E}, \mathcal{F})$ with positive kernel $\varkappa = i \circ^t j$ where $i : H \longrightarrow \mathcal{E}$ and $j = \bar{i} : \overline{H} \longrightarrow \bar{\mathcal{E}} = \mathcal{E}$ are the canonical injections. Its transpose ${}^t\varkappa = j \circ^t i = \overline{\varkappa}$ is the kernel of the subduality $(\overline{H}, H)$.

From the isomorphism between $\mathbf{L}((\mathbb{K}^\Omega)', \mathbb{K}^\Omega)$ and $\mathbb{K}^{\Omega\times\Omega}$, the kernel of evaluation dualities can be identified with a unique kernel function that holds numerous properties. From this identification, we also call evaluation dualities reproducing kernel dualities.

Definition 1.13 (– reproducing kernel –)
Let (E, F) be an evaluation duality of Ω with kernel $\varkappa$.
We call reproducing kernel (function) of (E, F) the function of two variables:

$$\begin{array}{rcl} K : \Omega \times \Omega & \longrightarrow & \mathbb{K} \\ t, s & \longmapsto & K(t,s) = ({}^t\varkappa(\delta_s), \varkappa(\delta_t))_{(F,E)} \end{array}$$

Conversely, the kernel $\varkappa$ can be easily deduced from K by the relation

$$\varkappa(\delta_t) = K(t,.)$$

We deduce from this the following reproduction formulas for the kernel function:

Corollary 1.14

1. $\forall s \in \Omega, \forall e \in E,\ e(s) = (K(.,s), e)_{(F,E)}$

2. $\forall t \in \Omega, \forall f \in F,\ f(t) = (f, K(t,.))_{(F,E)}$

3. $K(t,s) = (K(.,s), K(t,.))_{(F,E)}$.

Proof Let us prove the second assertion. We apply Theorem 1.11:

$$\begin{aligned}\forall f \in F, t \in \Omega,\ f(t) &= (\delta_t, j(f))_{((\mathbb{K}^\Omega)', \mathbb{K}^\Omega)} \\ &= L(f, \varkappa(\delta_t)) \text{ from Theorem 1.11} \\ &= L(f, K(t,.))\end{aligned}$$

The last assertion is just the previous formula with $f(.) = K(.,s)$.

example 1 **Polynomials, splines**
The kernel of the subduality $(E_{\mathcal{P}}, F_{\mathcal{P}})$ of $\mathbb{R}^{\mathbb{R}}$ is identified with the kernel function

$$K_{\mathcal{P}}(t,s) = (t-s)^n$$

Remark that when n is odd this kernel is antisymmetric.

example 2 **Entire functions and Hermite polynomials**
We have previously seen that the reproducing kernel of the evaluation duality (E_H, F_H) is the two-variable function

$$K_H(z,w) = \sum_{n \in \mathbb{N}} \frac{H_n(w) z^n}{n!} = e^{-z^2 + 2zw}$$

It is the generating function of the Hermite polynomials.

example 3 **Harmonic and Hyperharmonic functions**
We have seen that the duality (E_m, F_m) is an evaluation duality on B and that there exists a two-variable function kernel function $K_m(x,y)$ on B verifying:

$$\forall f \in E_m, \quad f(y) = \int_B K_m(x,y) f(x) d\nu_m(x), \quad y \in B$$

$$\forall g \in F_m, \quad g(x) = \int_B K_m(x,y) g(y) d\nu_m(y), \quad x \in B$$

with

$$\forall x \in B,\ K_m(x,.) \in E_m \text{ and } \forall y \in B,\ K_m(.,y) \in F_m$$

By unicity of the kernel function, we deduce that K_m is the reproducing kernel of (E_m, F_m).

1.5 The range of the kernel: the primary subduality

The image (or range) of a positive kernel plays a special role in the theory of Hilbert subspaces: it is a prehilbertian subspace dense in the Hilbert subspace, that is actually its completion. This latter point cannot be attained for the moment due to the too big generality of subdualities. That will however be the crucial point in the section 3 "canonical subdualities".
However, the two other points remain for any kernel as we will see below.

Definition 1.15 *We call* <u>*primary subduality*</u> *associated to a kernel $\varkappa$ the subspaces of $\mathcal{E}$ $E_0 = \varkappa(\mathcal{F})$ and $F_0 =^t \varkappa(\mathcal{F})$ put in duality by the following bilinear form L_0:*

$$\begin{array}{rcl} L_0 : F_0 \times E_0 & \longrightarrow & \mathbb{K} \\ ({}^t\varkappa(\varphi_1), \varkappa(\varphi_2)) & \longmapsto & (\varphi_1, \varkappa(\varphi_2))_{(\mathcal{F},\mathcal{E})} = ({}^t\varkappa(\varphi_1), \varphi_2)_{(\mathcal{E},\mathcal{F})} \end{array}$$

Remark that the bilinear form is well defined since the elements of $\ker(\varkappa)$ are orthogonal to ${}^t\varkappa(\mathcal{F})$ and respectively, the elements of $\ker({}^t\varkappa)$ are orthogonal to $\varkappa(\mathcal{F})$.

Lemma 1.16 *The primary subduality is a subduality of $(\mathcal{E}, \mathcal{F})$. Its kernel is $\varkappa$. Any kernel may then be associated to at least one subduality.*

Proof From the definition of the primary duality we verify easily that

- $E_0 \subseteq \mathcal{E}, \;\; F_0 \subseteq \mathcal{E};$
- $\gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|E_0}) \subseteq \gamma_{(E_0,F_0)}(F_0), \;\; \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|F_0}) \subseteq \gamma_{(F_0,E_0)}(E_0).$

and from the definition of L_0 that its kernel is $\varkappa$.

The primary subduality of a reproducing kernel duality is simply

$$E_0 = \{e = \sum_{i=1}^{n} \alpha_i K(t_i, .), \; n \in \mathbb{N}, \, \alpha_i \in \mathbb{K}, \, t_i \in \Omega\}$$

$$F_0 = \{f = \sum_{j=1}^{m} \beta_i K(., s_i), \; m \in \mathbb{N}, \, \beta_j \in \mathbb{K}, \, s_j \in \Omega\}$$

with bilinear form

$$(f, e)_{(F_0,E_0)} = \sum_{1 \le i \le n, 1 \le j \le m} \alpha_i \beta_j K(t_i, s_j)$$

The following theorem gives an interesting result of denseness:

Theorem 1.17 *Let (E, F) be a subduality with kernel $\varkappa$. Then the primary subduality (E_0, F_0) associated to $\varkappa$ is dense in (E, F) for any topology compatible with the duality.*

Proof We use Corollary p 109 [20]: "If $u : E \longrightarrow \mathcal{E}$ is one-to-one, its transpose ${}^t u : \mathcal{E}' \longrightarrow E'$ has weakly dense image". Equivalently its transpose considering dual systems ${}^t u : \mathcal{F} \longrightarrow F$ has weakly dense image. Taking $u = j$ gives the desired result since there is an equivalence between closure and weak closure for convex sets (and E_0 is convex), Theorem 4 p 79 [20].

It follows that the primary subduality associated with $\varkappa$ may be seen as the smallest subduality (in terms of inclusion) of $(\mathcal{E}, \mathcal{F})$ with kernel $\varkappa$.

The kernel then defines almost completely the bilinear form, precisely:

Proposition 1.18 *Let (E, F) and (H, R) be two subdualities with the same kernel $\varkappa$. Then*

$$\forall \psi \in F \cap R, \forall \varphi \in E \cap H, \qquad (\psi, \varphi)_{(F,E)} = (\psi, \varphi)_{(R,H)}$$

Proof The topology on E, F, H and R is the weak topology. Let us endow the vector space $F \cap R$ (resp. $E \cap H$) with the projective limit topology with respect to the canonical inclusions $\mathfrak{f} : F \cap R \longrightarrow F$, $\mathfrak{r} : F \cap R \longrightarrow R$ (resp. $\mathfrak{e} : E \cap H \longrightarrow E$, $\mathfrak{h} : E \cap H \longrightarrow H$), *i.e.* the coarsest topology that makes these inclusions continuous. $F \cap R$ (resp. $E \cap H$) is then a locally convex vector space ([11, II, Proposition 4 p. 29]).
Define the following bilinear form on $(F \cap R) \times (E \cap H)$:

$$\begin{array}{rcl} \mathbf{B} : (F \cap R) \times (E \cap H) & \longrightarrow & \mathbb{K} \\ (\psi \, , \, \varphi) & \longmapsto & (\psi, \varphi)_{(F,E)} - (\psi, \varphi)_{(R,H)} \end{array}$$

This bilinear form is separately continuous by composition of continuous applications. Evaluating this bilinear form on ${}^t\varkappa(\mathcal{F}) \times \varkappa(\mathcal{F})$ we get

$$\mathbf{B}\left({}^t\varkappa(\mathcal{F}) \times \varkappa(\mathcal{F})\right) = 0$$

since the two subdualities have the same kernel $\varkappa$.
But ${}^t\varkappa(\mathcal{F})$ is dense in $F \cap R$ (resp. $\varkappa(\mathcal{F})$ in $E \cap H$) by [20, Proposition 1 p. 2] (recall that it is weakly dense in F and R by Theorem 1.17). It follows that $\mathbf{B}$ is null on $(F \cap R) \times (E \cap H)$ [11, III, Proposition 7 p. 32].

The following corollary is straightforward:

Corollary 1.19 *under the previous assumptions,* $(E \cap H, F \cap R)$ *endowed with the previous bilinear form is a subduality of* $(\mathcal{E}, \mathcal{F})$ *with kernel* $\varkappa$*.*

2 Effect of a weakly continuous linear application and algebraic structure of $\mathcal{SD}((\mathcal{E}, \mathcal{F}))$

We have defined the set of subdualities. It is of prime interest to know what operations one can perform on this set and particularly if one can endow this set with the structure of a vector space. This can be attained by first studying the effect of a weakly continuous linear application.

2.1 Effect of a weakly continuous linear application

We suppose now we are given a second pair of spaces in duality $(\mathfrak{E}, \mathfrak{F})$. It is actually possible to define the image subduality by a weakly continuous linear application $u : \mathcal{E} \to \mathfrak{E}$, of a subduality (E, F) of $(\mathcal{E}, \mathcal{F})$, by using orthogonal relations in the duality (E, F).
$\forall A \subset \mathcal{E}$, $u_{|A}$ denotes the restriction of u to the set A. We then define the following quotient spaces:

$$\mathcal{M} = \left(\ker(u_{|F})^{\perp} / \ker(u_{|E})\right) \text{ and } \mathcal{N} = \left(\ker(u_{|E})^{\perp} / \ker(u_{|F})\right)$$

Lemma 2.1 *The linear applications* $u_{|\mathcal{M}}$ *and* $u_{|\mathcal{N}}$ *are well defined and injective, and* $\forall\, (\dot{m}, \dot{n}) \in \mathcal{M} \times \mathcal{N}$*, the bilinear form* $B(u_{|\mathcal{N}}(\dot{n}), u_{|\mathcal{M}}(\dot{m})) = (n, m)_{(F,E)}$ *defines a separate duality* $(u_{|\mathcal{M}}(\mathcal{M}), u_{|\mathcal{N}}(\mathcal{N}))$*.*

Proof We have the following factorisation

$$u : \ker(u_{|F})^{\perp} \longrightarrow (\ker(u_{|F})^{\perp} / \ker(u_{|E}) \stackrel{u_{|\mathcal{M}}}{\longrightarrow} \mathfrak{E}$$

and $u_{|\mathcal{M}}$ (resp. $u_{|\mathcal{N}}$) is one-to-one. Moreover the bilinear form $B : u_{|\mathcal{M}}(\mathcal{M}) \times u_{|\mathcal{N}}(\mathcal{N}) \longrightarrow \mathbb{K}$ is well defined since:
$\forall (m_1, m_2) \in \dot{m}, \forall (n_1, n_2) \in \dot{n},\ (m_1 - m_2, n_1 - n_2)_{(E,F)} = 0.$

The definition of the subduality image of (E, F) by u is then included in the following theorem:

Theorem 2.2 (– subduality image –)
The duality $(u_{|\mathcal{M}}(\mathcal{M}), u_{|\mathcal{N}}(\mathcal{N}))$ *is a subduality of* $(\mathfrak{E}, \mathfrak{F})$ *called* <u>*subduality image*</u> *of* (E, F) *by* u *and denoted* $u((E, F))$*. Its kernel is* $u \circ \varkappa \circ^t u$*.*

Proof The algebraic inclusions of definition 1.2 are fulfilled and the dual system $(u_{|\mathcal{M}}(\mathcal{M}), u_{|\mathcal{N}}(\mathcal{N}))$ is a subduality of $\mathfrak{E}$.
Let $\tilde{i} : u_{|\mathcal{M}}(\mathcal{M}) \longrightarrow \mathfrak{E}$ and $\tilde{j} : u_{|\mathcal{N}}(\mathcal{N}) \longrightarrow \mathfrak{E}$ be the canonical inclusions. $u \circ \varkappa \circ^t u$ satisfies the requirements of Theorem 1.11 since:

$$\forall n \in u_{|\mathcal{N}}(\mathcal{N}), \forall \mathfrak{f} \in \mathfrak{F},\ \left(\mathfrak{f}, \tilde{j}(n)\right)_{(\mathfrak{F},\mathfrak{E})} = B\left(n, \tilde{i}^{-1} \circ u \circ \varkappa \circ^t u(\mathfrak{f})\right)$$

Let f an antecedent by u of n in F. Then:

$$\begin{aligned} B\left(n, \tilde{i}^{-1} \circ u \circ \varkappa \circ^t u(\mathfrak{f})\right) &= (f, \varkappa \circ^t u(\mathfrak{f}))_{(F,E)} \\ &= (f,^t u(\mathfrak{f}))_{(\mathcal{E},\mathcal{F})} \\ &= (u(f), \mathfrak{f})_{(\mathfrak{E},\mathfrak{F})} \\ &= \left(\mathfrak{f}, \tilde{j}(n)\right)_{(\mathfrak{F},\mathfrak{E})} \end{aligned}$$

We conclude by unicity of the kernel.

Remark that the subduality image $u((E,F))$ is included in the set $(u(E), u(F))$ but smaller in general.

We have the following figure :

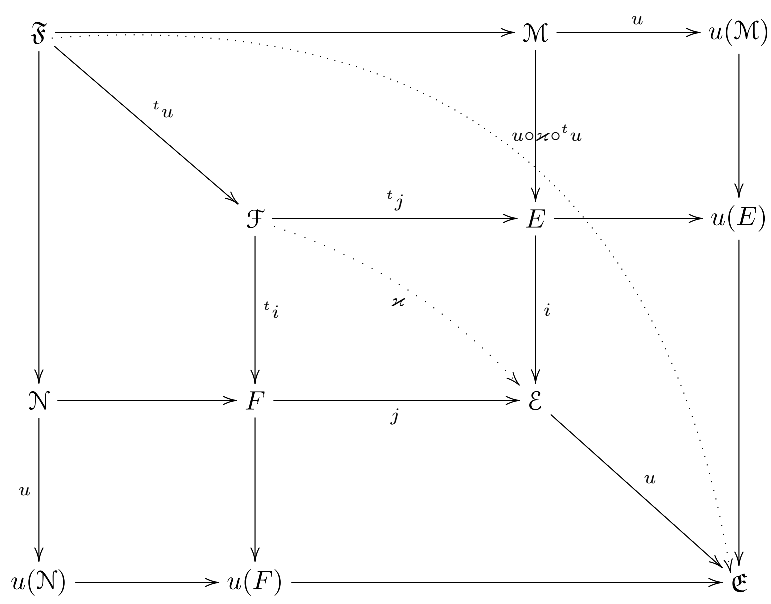

Figure 3: Subduality image

example 1 **Restriction of evaluation dualities**

Let Ω be any set and $\Theta \subset \Omega$. Let

$$\begin{array}{rcl} \theta : \mathbb{K}^\Omega & \longrightarrow & \mathbb{K}^\Theta \\ \phi & \longmapsto & \phi_{|\Theta} \end{array}$$

be the operator of restriction to Θ and let $(E,F) \hookrightarrow \mathbb{K}^\Omega$.
What is $\theta((E,F))$?
Using our definition, we get that $\theta((E,F)) = (H,R)$

$$H = \{e_{|\Theta},\ e \in E,\ f_{|\Theta} = 0 \Rightarrow (f,e)_{(F,E)} = 0\}$$

$$R = \{f_{|\Theta},\ f \in F,\ e_{|\Theta} = 0 \Rightarrow (f,e)_{(F,E)} = 0\}$$

with duality product $\left(f_{|\Theta}, e_{|\Theta}\right)_{(R,H)} = (f,e)_{(F,E)}$
Remark that $H \neq E_{|\Theta}$ and $R \neq F_{|\Theta}$ in general.
$\theta((E,F))$ admits for kernel function $K_{|\Theta\times\Theta}$.

It is worth noticing that the transport of structure is the basic tool for the construction of subdualities.

2.2 The vector space $(\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi),+,*)$

Suppose we are given two dual systems $(\mathcal{E}_1,\mathcal{F}_1)$ and $(\mathcal{E}_2,\mathcal{F}_2)$ and two subdualities $(E_1,F_1) \subset \mathcal{SD}((\mathcal{E}_1,\mathcal{F}_1))$ and $(E_2,F_2) \subset \mathcal{SD}((\mathcal{E}_2,\mathcal{F}_2))$. Then it is straightforward to see that the direct product $(E_1 \times E_2, F_1 \times F_2)$ endowed with the canonical bilinear form is a subduality of $(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F}_1 \times \mathcal{F}_2)$. Theorem 2.2 then allows us to define the operations of addition and external multiplication on the set $\mathcal{SD}((\mathcal{E},\mathcal{F}))$ by considering the weakly continuous morphisms $+ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and $* : \mathbb{K} \times \mathcal{E} \to \mathcal{E}$. The associated operations for the kernels are then addition and external multiplication on $\mathbf{L}(\mathcal{F},\mathcal{E})$.
However the addition is not associative:

$$\{(E_1,F_1) - (E_1,F_2) = 0\} \nRightarrow \{(E_1,F_1) = (E_2,F_2)\}$$

hence

$$((E_1,F_1) + (E_1,F_2)) + (E_3,F_3) \neq (E_1,F_1) + ((E_2,F_2) + (E_3,F_3))$$

in general and $(\mathcal{SD}((\mathcal{E},\mathcal{F})),+)$ is only a magma. Remark that this peculiar situation was already embarrassing when dealing with Hermitian subspaces, as noted by Schwartz [37].

In order to define a vector space structure appears the necessity of the following equivalence relation (induced by $\ker(\Phi)$):

$$(E_1,F_1)\mathcal{R}(E_2,F_2) \iff (E_1,F_1) - (E_2,F_2) = 0 \iff \varkappa_1 = \varkappa_2$$

Theorem 2.3 *The set* $(\mathcal{SD}((\mathcal{E},\mathcal{F})),+,*)$ *is a commutative unital magma for* $+$ *where every element admits a (non necessarily unique) symmetric. The extern multiplication is distributive over the addition.*
The set $(\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi),+,*)$ *is a vector space over* $\mathbb{K}$ *algebraically isomorphic to the vector space of kernels* $\mathbf{L}(\mathcal{F},\mathcal{E})$*, an isomorphism being*

$$\Phi : \mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi) \longrightarrow \mathbf{L}(\mathcal{F},\mathcal{E})$$

Proof The following relation

$$(E_1,F_1)\mathcal{R}(E_2,F_2) \iff (E_1,F_1)-(E_2,F_2)=0 \iff \varkappa_1=\varkappa_2$$

is an equivalence relation and the quotient set $\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi)$ is in bijection with the set of kernels $\mathbf{L}(\mathcal{F},\mathcal{E})$.
One verifies rapidly that the addition and external multiplication are compatible with this bijection, which gives the vector space structure of the set $\mathcal{SD}(\mathcal{E})/\ker(\Phi)$ and the isomorphism of vector space between $\mathcal{SD}(\mathcal{E})/\ker(\Phi)$ and $\mathbf{L}(\mathcal{F},\mathcal{E})$.

example 1 **Polynomials, splines**
For $k\in[0,n]$ define the following one-dimensional evaluation duality with reproducing kernel $K_k(t,s)=C_n^k t^{n-k}(-s)^k$,

$$E_k=\mathbb{R}.s^k, \qquad F_k=\mathbb{R}.t^{n-k}$$

with duality product:

$$\left(x^{n-k},x^k\right)_{(F_k,E_k)}=\frac{(-1)^k}{C_n^k}$$

We can give a sense to the sum (either by associativity of this particular sum, of by the image of the operator n-sum)

$$(E,F)=\sum_{k=0}^{n}(E_k,F_k)$$

It is the (unique since finite-dimensional) subduality with kernel

$$K(t,s)=\sum_{k=0}^{n}C_n^k t^{n-k}(-s)^k=(t-s)^n=K_{\mathcal{P}}(t,s)$$

that is $(E,F)=(E_{\mathcal{P}},F_{\mathcal{P}})$.

example 2 $+$ **is not associative on** $\mathcal{SD}((\mathcal{E},\mathcal{F}))$
Let $(E,F)\hookrightarrow(\mathcal{E},\mathcal{F})$ be different from its primary subduality (E_0,F_0). Then

$$(E_0,F_0)-(E,F)=(0,0)$$

since its kernel is the null operator. It follows that

$$((E_0, F_0) - (E, F)) + (E, F) = (E, F)$$

and

$$(E_0, F_0) + (-(E, F) + (E, F)) = (E_0, F_0)$$

that are different by hypothesis. They are of course in the same equivalence class for they have the same kernel.

It is also possible to give proper definitions of infinite sums and integrals of subdualities. It is the object of a forthcoming paper that will develop a theory of harmonic analysis on subdualities.

2.3 Categories and functors

Let $\mathcal{C}$ the category of dual systems $(\mathcal{E}, \mathcal{F})$ the morphisms being the weakly continuous linear applications and $\mathcal{V}$ the category of vector spaces the morphisms being the linear applications. Then according that to a morphism $u : \mathcal{E} \longrightarrow \mathfrak{E}$ we associate the morphism

$$\begin{array}{rcl} \tilde{u} : \mathcal{SD}((\mathcal{E}, \mathcal{F}))/\ker(\Phi) & \longrightarrow & \mathcal{SD}((\mathfrak{E}, \mathfrak{F}))/\ker(\Phi) \\ (E \dot{,} F) & \longmapsto & u((E \dot{,} F)) \end{array}$$

we get

Theorem 2.4 $\frac{\mathcal{SD}}{\ker(\Phi)} : (\mathcal{E}, \mathcal{F}) \mapsto \mathcal{SD}((\mathcal{E}, \mathcal{F}))/\ker(\Phi)$ *is a covariant functor of category* $\mathcal{C}$ *into category* $\mathcal{V}$.

On the other hand, $\mathbf{L} : (\mathcal{E}, \mathcal{F}) \mapsto \mathbf{L}(\mathcal{F}, \mathcal{E})$ is also a covariant functor of category $\mathcal{C}$ into category $\mathcal{V}$, according that to a morphism $u : \mathcal{E} \longrightarrow \mathfrak{E}$ we associate the morphism

$$\begin{array}{rcl} \tilde{u} : \mathbf{L}(\mathcal{F}, \mathcal{E}) & \longrightarrow & \mathbf{L}(\mathfrak{F}, \mathfrak{E}) \\ \varkappa & \longmapsto & u \circ \varkappa \circ^t u \end{array}$$

and

Theorem 2.5 *The two covariant functors* $\frac{\mathcal{SD}}{\ker(\Phi)}$ *and* $\mathbf{L}$ *are isomorphic.*

3 Canonical subdualities

The classes of equivalence of subdualities with identical kernel are very large and it may be interesting to associate each equivalence class with a canonical representative enjoying good properties, as it was done for positive kernels associated to a unique Hilbert subspaces. This section aims at defining this particular set of subdualities that will be called canonical subdualities. The desired good properties (such that the equality with Hilbert subspaces in case of positive kernels) are listed below.

Actually, before stating the main results of this part, one must ask the following question: what do we mean by canonical representative? And what good properties do we need?
There is probably not a single answer to these questions and there may be many different good ways to define canonical representatives. However, it seems natural to require some properties for a canonical representative. Those chosen here are:

1. the canonical representative must be "representative" of the kernel, i.e. entirely defined by the kernel;
2. the canonical representative must be "big", in some sense;
3. the definition of the canonical representative must be "symmetric", i.e. if (E, F) is the canonical subduality associated to $\varkappa$, then (F, E) must be the canonical subduality associated to ${}^t\varkappa$;
4. the definition of the canonical representative must coincide with the definition of real Hilbert subspace in case of (real) positive kernels.

It is in this spirit that those canonical subdualities have been constructed.

Since Hilbert subspaces may be seen as the completion of the primary subspace associated to the positive kernel it seems natural to mimic this construction up to a certain extent i.e. do some completion. However, in the general case there is no canonical norm (or equivalently canonical unit ball) associated to the kernel. The first task is then to define "canonical" topologies on the sets E and F.

3.1 Definition of the canonical topologies

We define the locally convex topology by convergence on bounded sets of a dual space. First we aim at defining some "good" bounded sets. Our choice is as

follows:
Let $\varkappa \in \mathbf{L}(\mathcal{F},\mathcal{E})$ be a kernel, (E_0, F_0) the associated primary subduality. We recall that a barrel is a closed, equilibrated and absorbing set. We define the following sets:

- $\mathcal{T}_{E_0} = \Big\{\sigma$ barrels of $E_0,\ \exists(\lambda,\gamma) \in (\mathbb{R}^+)^2,\ \Re\big(\big(\varkappa^{-1}(\sigma),\sigma\big)_{(\mathcal{F},\mathcal{E})}\big) \leq \lambda$ and $\Re\big(\big({}^t\varkappa^{-1}(\sigma^\circ),\sigma^\circ\big)_{(\mathcal{F},\mathcal{E})}\big) \leq \gamma\Big\}$
  where σ° is the polar (remark that since we deal with barrels, the polar coincide with the absolute polar) of σ for the duality (E_0, F_0);
- $\mathcal{T}_{F_0} = \{\sigma^\circ,\ \sigma \in \mathcal{T}_{E_0}\}$;

under the following convention:
$\Re\big(\big(\varkappa^{-1}(\sigma),\sigma\big)_{(\mathcal{F},\mathcal{E})}\big) \leq \lambda$ stands for $\exists\varsigma \in \mathcal{F},\ \varkappa(\varsigma) = \sigma$ and $\Re\big((\varsigma,\sigma)_{(\mathcal{F},\mathcal{E})}\big) \leq \lambda$ (resp. for σ°).

Remark that this convention is useless for symmetric, Hermitian or antisymmetric kernels since $\ker(\varkappa)$ (resp. $\ker({}^t\varkappa)$) is orthogonal to ${}^t\varkappa(\mathcal{F})$ (resp. to $\varkappa(\mathcal{F})$) and obviously if the kernel $\varkappa$ is one-to-one.

$\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) is a set of weakly bounded sets of (E_0, F_0) and one can define over F_0 (resp. E_0) the topology of $\mathcal{T}_{E_0}$-convergence, this topology being locally convex and compatible with the vector space structure (Proposition 16 p. 86 [20]).

Let us show that $\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) is a set of weakly bounded sets:
Let $\sigma \in \mathcal{T}_{E_0}$. It is an equilibrated and absorbing set hence $\forall f \in F, \exists\alpha > 0, \alpha f \in \mathcal{T}_{E_0}$ and $(\sigma, f)_{(E,F)}$ is bounded. It follows that $\sigma^\circ \in \mathcal{T}_{F_0}$ is a barrel as the absolute polar of an equilibrated weakly bounded set (Corollary 3 p 68 [10]) and finally, the elements of $\mathcal{T}_{F_0}$ are also weakly bounded.

3.2 Construction of the canonical subdualities

We cannot start from any kernels and therefore restrict our attention to a subset of kernels that we call stable kernels:

Definition 3.1 *Let $\varkappa \in \mathbf{L}(\mathcal{F},\mathcal{E})$ a kernel. It is stable if:*

1. *the sets $\mathcal{T}_{E_0}$ and $\mathcal{T}_{F_0}$ are non empty;*

2. *$\varkappa : \mathcal{F} \longrightarrow E_0$ (resp. $\varkappa : \mathcal{F} \longrightarrow F_0$) is continuous if $\mathcal{F}$ is endowed with the Mackey topology and E_0 with the topology of $\mathcal{T}_{F_0}$-convergence (resp. F_0 with the $\mathcal{T}_{E_0}$-convergence).*

The first condition is necessary to be able to define the canonical topologies whereas the second condition is needed to perform the completion (see Lemma 3.3 below).

Proposition 3.2 *The second condition is equivalent to:*
the elements of $\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) are weakly relatively compact in $\mathcal{E}$.
This condition is always fulfilled if $\mathcal{F}$ is (Mackey) barreled.

Proof We use Proposition 28 p 110 in [20]. The weakly continuous application $\varkappa =^t j : \mathcal{F} \longrightarrow E_0$ is continuous if $\mathcal{F}$ is endowed with the Mackey topology and E_0 with the topology of $\mathcal{T}_{F_0}$-convergence if and only if $j(\mathcal{T}_{F_0})$ is a set of weakly relatively compact sets of $\mathcal{E}$ (recall that the Mackey topology on $\mathcal{F}$ is the topology of convergence on the weakly compact sets of $\mathcal{E}$).

Lemma 3.3 *Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a stable kernel, (E_0, F_0) the associated primary duality. Let $E = \widehat{E_0}$ (resp. $F = \widehat{F_0}$) be the completion of E_0 endowed with the topology of $\mathcal{T}_{F_0}$-convergence (resp. the completion of F_0 endowed with the topology of $\mathcal{T}_{E_0}$-convergence). Then E (resp. F) is the vector space generated by the closures (in $\widehat{E_0}$, resp. $\widehat{F_0}$) of the convex envelopes of finite unions of elements of $\mathcal{T}_{E_0}$ (resp.$\mathcal{T}_{F_0}$) and $E \subset \mathcal{E},\ F \subset \mathcal{E}$.*

Proof First, $E = \widehat{E_0}$ is the vector space generated by the closures in $\widehat{E_0}$ of its neighborhoods of zero, i.e. by polarity by the closures of the convex envelopes of finite unions of elements of $\mathcal{T}_{E_0}$.
Second, if we endow $\mathcal{F}$ with the Mackey topology and F_0 with the $\mathcal{T}_{E_0}$-convergence, then $\varkappa : \mathcal{F} \longrightarrow F_0$ is continuous with dense image and $\varkappa : F_0' \longrightarrow \mathcal{E}$ is one-to-one. But F_0' is the vector space generated by the weak closures of the convex envelopes of finite unions of elements of $\mathcal{T}_{E_0}$ in the weak completion of E_0 (Corollary 1 p 91 [20]). It follows that $E \subset F_0' \subset \mathcal{E}$ since $\widehat{E_0}$ is continuously included in the weak completion of E_0.

Theorem 3.4 ($-$ canonical subduality $-$ **)**
Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a stable kernel, (E_0, F_0) the associated primary duality, E and F defined as before. Then the bilinear form L_0 defined on the primary duality extends to a unique bilinear form L on $F \times E$ separate. It defines a duality (E, F) called canonical subduality associated to $\varkappa$.

Proof We use the extension of bilinear hypocontinuous forms theorem (Proposition 8 p 41 [10]). We endow E (resp. F) with the topology of $\mathcal{T}_{F_0}$ (resp. $\mathcal{T}_{E_0}$)-convergence. Then E_0 (resp. F_0) is dense in E (resp. F), every point of E (resp. F) lies in the closure of an element of $\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) and $L_0 : F_0 \times E_0 \longrightarrow \mathbb{K}$ is hypocontinuous with respect to $\mathcal{T}_{E_0}$ and $\mathcal{T}_{F_0}$. The hypothesis of the theorem are then fulfilled and L_0 extends on a unique bilinear form L on $F \times E$. This form is separate by the Hahn-Banach theorem.

Remark 3.5 *L is hypocontinuous with respect to $\mathcal{T}_{F_0}$ and $\mathcal{T}_{E_0}$.*

3.3 Properties of canonical subdualities

In the introduction of this section, we ask for some properties of canonical subdualities. The following results prove that the constructed subduality holds indeed these properties.

Next corollary gives a important result concerning completeness of canonical subdualities:

Corollary 3.6 *If the elements of $\mathcal{T}_{E_0}$ (resp. of $\mathcal{T}_{F_0}$) are weakly relatively compacts in $\widehat{E_0}$ (resp. in $\widehat{F_0}$) then $E = \widehat{E_0}$ (resp. $F = \widehat{F_0}$) is complete for its Mackey topology.*

Proof The topology of $\mathcal{T}_{F_0}$-convergence (resp. of $\mathcal{T}_{E_0}$-convergence) is then compatible with the duality (E, F) and the result follows.

We call them weakly locally compact canonical subdualities, since the topologies of $\mathcal{T}_{E_0}$-convergence and of $\mathcal{T}_{F_0}$-convergence are weakly relatively compact. Respectively, a stable kernel verifying such conditions is called a weakly compact kernel.

Proposition 3.7

1. *if (E, F) is the canonical subduality associated to $\varkappa$, then (F, E) is the canonical subduality associated to ${}^t\varkappa$;*
2. *if $\varkappa$ is the Hilbert kernel of a real Hilbert subspace H, then $\varkappa$ is stable (weakly compact) and the associated canonical subduality is (H, H).*

Proof The first statement is obvious by construction and the second one is straightforward when dealing with real Hilbert spaces. It would not be the same for the field of complex numbers, since no conjugation in the definition of $\mathcal{T}_{H_0}$ is at stake.

Real Hilbert spaces give a very large breeding ground of canonical subdualities (different from Hilbertian ones in general) thanks to continuous coercive bilinear forms. Recall that a coercive bilinear form on a real Hilbert space H verifies:

$$\exists K > 0, B\,(h,h) \geq K \|h\|_H^2$$

The following proposition follows:

Proposition 3.8 *Let H be a real Hilbertian subspace of $(\mathcal{E},\mathcal{F})$ and B a continuous coercive bilinear form on H. Then H endowed with this bilinear form is a canonical subduality of $(\mathcal{E},\mathcal{F})$.*

Proof One checks easily that the convergence defining the canonical topologie takes place on the balls for the Hilbertian norm. By reflexivity of Hilbert spaces, the canonical topology is the Hilbertian one and we get that the duality (H,H) with bilinear form B is canonical.

example 1 **Sobolev spaces**

In this example, $\mathbb{K} = \mathbb{R}$. Then the subduality (E_W, F_W) is canonical. This is a direct consequence of the following results:

1. Let $\sigma \in \mathcal{T}_{E_0}$, $\left(\varkappa_W^{-1}(\sigma), \sigma\right)_{(\mathcal{F},\mathcal{E})} \leq \lambda$ and $\left({}^t\varkappa_W^{-1}(\sigma^\circ), \sigma^\circ\right)_{(\mathcal{F},\mathcal{E})} \leq \gamma$. Then
$$e(s) = \int_0^s \phi(t)dt \in \sigma \Rightarrow \int_\Omega \phi^2 \leq \lambda$$
and
$$f(t) = \int_t^1 \psi(s)ds \in \sigma^\circ \Rightarrow \int_\Omega \psi^2 \leq \gamma$$
2. By Schwartz inequality
$$B\lambda = \left\{e \in D',\ e(s) = \int_\Omega 1\!\!1_{t\leq s}\phi(t)dt,\ \int_\Omega \phi^2 \leq \lambda\right\} \in \mathcal{T}_{E_0}$$
3. The canonical topologies are then Hilbertian topologies and the completions are the given Sobolev spaces.

example 2 **Kreĭn subspaces**

Let $\varkappa$ be any real Hermitian kernel that admits a Kolmogorov decomposition. Then the canonical subduality associated to $\varkappa$ is the self-duality intersection of all Kreĭn subspaces with kernel $\varkappa$ (this intersection is well defined by Corollary 1.19).

example 3 **Symplectic Banach space**
Let B be a reflexive Banach space, B' its dual space. define

$$\begin{array}{rcl} \varkappa : (B' \times B) & \longrightarrow & (B \times B') \\ (b', b) & \longmapsto & (b, -b') \end{array}$$

First, notice that the kernel is stable since a Banach space is barreled and the unit ball is in $\mathcal{T}_{E_0}$. It follows that this kernel admits a canonical subduality. But the primary subduality $((B \times B'), (B \times B'))$ endowed with the symplectic bilinear form

$$\left(\left(b_f, b_f^{'}\right), \left(b_e, b_e^{'}\right)\right) = b_e b_f^{'} - b_e^{'} b_f$$

is the only subduality with kernel $\varkappa$ since the kernel is bijective. It is then the canonical subduality of $(B \times B')$ with kernel $\varkappa$.

3.4 The set of canonical subdualities

In chapter 2 the image of a subduality by a weakly continuous morphism has been defined. It is then of prime interest to see whether the image of a canonical subduality is a canonical subduality,. Actually, the main results of this section are of negative type:

- the set of canonical subdualities is not stable by the action of a weakly continuous linear application
- the set of canonical subdualities cannot be endowed with the structure of a vector space.

The second statement is evident by taking a real Kreĭn subspace of multiplicity. It hence defines no canonical subduality but it is the difference of two real Hilbert (hence canonical) subspaces.
For the first statement, the same argument works. A real Kreĭn subspace H of $\mathcal{E}$ of multiplicity is no canonical subspace, but it is the image by the canonical injection $i : H \longrightarrow \mathcal{E}$ of the canonical subduality (H, H) of (H, H).

We can then ask the following questions:

- Is it interesting to work with one canonical subduality, or should we keep many (if not all) “representatives” ?
- Are there particular dualities such that the image of any of their canonical subdualities is canonical (different from finite-dimensional ones) ?

- Conversely, what are the kernels such that the image of their canonical subdualities is canonical (apart from positive kernels or finite-dimensional ones) ?

One must however notice from the counterexamples of this section that our choice of canonical subdualities is not important, for as soon as we have Hilbertian subspaces and their difference, no definition of canonical subduality will give a set stable by sum or image.

4 Applications

In this section we detail three different possible applications of this theory:

1. the first one is the study of normal subdualities and the associated concept of Green operators, which is a continuation of L. Schwartz work on normal Hilbert subspaces ([37]) and that could be applied to many problems in differential equations or other topics (see [29]).
2. The second one considers group representation in locally convex spaces and invariant subdualities The idea is that a general theory of harmonic analysis on subdualities is possible. In particular we search the subdualities of holomorphic functions invariant under the action of the group of similitudes.
3. Finally a third study is the generalization of the Berezin symbol for operator in evaluation dualities, where once again the special case of holomorphic functions is of interest.

4.1 Normal subdualities and Green operators

The Green function associated to a differential operator is a classical tool in differential analysis, but a precise definition of the Green function is only given for positive differential operators in [37]. In this section we give a rigorous definition of the Green operator associated to a kernel when the kernel is normal that generalizes L. Schwartz's definition and transform an algebraic problem -the existence of an inverse- into a topological problem -being a normal subduality-.

From now on we suppose that we are given a continuous injection u from $\mathcal{F}$ to $\mathcal{E}$, such that $\mathcal{F}$ is identified with a dense subspace of $\mathcal{E}$. (The classical example is the identification of the test functions as distributions).

Let now (E, F) be a subduality of $(\mathcal{E}, \mathcal{F})$. Then we say that this subduality is normal if $\mathcal{F}$ is identified with a dense subspace of E and F:

Definition 4.1 (– normal subduality –)
With the previous notations, (E, F) *subduality of* $(\mathcal{E}, \mathcal{F})$ *is* <u>*normal*</u> *if*

1. $u(\mathcal{F})$ *is dense in* E *and* F *for their Mackey topology;*
2. *the injection* u_E *from* $\mathcal{F}$ *to* E *is weakly continuous;*
3. *the injection* u_F *from* $\mathcal{F}$ *to* F *is weakly continuous.*

A kernel $\varkappa$ will be normal if there exists a normal subduality with kernel $\varkappa$.

The definition of a normal subspace is an old concept, see [37].

But we may consider $\theta_{(\mathcal{F},\mathcal{E})} \circ^t u_E$ and $\theta_{(\mathcal{F},\mathcal{E})} \circ^t u_F$ as canonical inclusions *i.e.* identifie for instance $f' \in F'$ with the unique element of $\mathcal{E}$ defining on $\mathcal{F}$ the continuous linear form $\varphi \longmapsto (f', \varphi)_{(F',F)}$.

$$\forall \varphi \in \mathcal{F},\ (\varphi, f')_{\mathcal{F},\mathcal{E})} = (f', \varphi)_{(F',F)}$$

It follows that with these identifications, (F', E') is a subduality of $(\mathcal{F}, \mathcal{E})$ with kernel $\mathrm{G} = \theta_{(\mathcal{F},\mathcal{E})} \circ^t u_F \circ \gamma_{(F,E)} \circ u_E$ (figure 4). Moreover, this subduality is also normal.

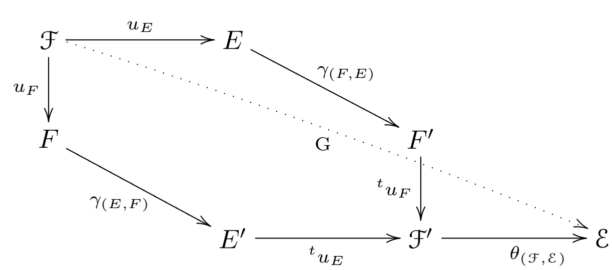

Figure 4: Illustration of a normal subduality and the relative inclusions

The bilinear form is given by:

$$(e', f')_{(E',F')} = (e', e)_{(E',E)}$$

where $e \in E$ verifies:

$$\forall f \in F, \quad (f, e)_{(F,E)} = (f', f)_{(F',F)}$$

If moreover $f \in \mathcal{F}$ we get

$$(f, e)_{(F,E)} = (f', f)_{(F',F)} = (f, f')_{(\mathcal{F},\mathcal{E})}$$

We can then state the following theorem:

Theorem 4.2 *$\varkappa$ extends to a continuous linear application from F' to E,* G *extends to a continuous linear application from E to F' and the two are inverse one from another.*

Proof The desired extensions are respectively $\theta_{(F,E)}$ and $\gamma_{(F,E)}$ which finishes the proof.

Suppose now we are given a normal kernel $\varkappa$. We have seen that G may be considered as the inverse of $\varkappa$. Is the operator G unique ? That is starting from two different normal subdualities with kernel $\varkappa$, do their dual spaces have the same kernel ? The answer is indeed positive:

Theorem 4.3 *Let (E, F) be a normal subduality with kernel $\varkappa$,* G *the kernel of (F', E'). Then any normal subduality (H, R) with kernel $\varkappa$ verifies that* G *is the kernel of (R', H').*

Proof To simplify the notations, we forget the canonical inclusions. We want to prove that G is the kernel of (R', H'), *i.e.*

$$\forall h' \in H', \forall \varphi \in \mathcal{F},\ (\varphi, h')_{(\mathcal{F},\mathcal{E})} = (h', \mathrm{G}(\varphi))_{(H',R')}$$

but by identification

$$(\varphi, h')_{(\mathcal{F},\mathcal{E})} = (h', \varphi)_{(H',H)}$$

and by definition of the bilinear form on (R', H')

$$(h', \varphi)_{(H',H)} = (h', r')_{(H',R')}$$

where $r' \in R'$ verifies

$$\forall \rho \in R, \quad (\rho, \varphi)_{(R,H)} = (r', \rho)_{(R',R)}$$

By the Hahn-Banach theorem it is sufficient to prove this equality on a dense subset of R, for instance $\mathcal{F}$. The problem reduces to prove that

$$\forall \psi \in \mathcal{F}, \quad (\psi, \varphi)_{(R,H)} = (\psi, \mathrm{G}(\varphi))_{(\mathcal{F},\mathcal{E})}$$

But since G is actually the kernel of (E', F') the same chain of reasoning gives that

$$\forall \psi \in \mathcal{F},\ \forall \varphi \in \mathcal{F}, \quad (\psi, \varphi)_{(F,E)} = (\psi, \mathrm{G}(\varphi))_{(\mathcal{F},\mathcal{E})}$$

We then use Proposition 1.18 and the fact that $\mathcal{F}$ is included in R, H, F and E by definition of a normal subduality.
It follows that

$$\forall \psi \in \mathcal{F},\ \forall \varphi \in \mathcal{F}, \quad (\psi, \varphi)_{(R,H)} = (\psi, \varphi)_{(F,E)} = (\psi, \mathrm{G}(\varphi))_{(\mathcal{F},\mathcal{E})}$$

and G is the kernel of (R', H').

The definition of the Green operator of a normal kernel follows:

Definition 4.4 (– Green operator –)
We call Green operator *of a normal kernel* $\varkappa$ *the kernel* G *of* (F', E') *where* (E, F) *is any normal subduality with kernel* $\varkappa$*.*

From Theorem 4.2 the Green operator G of $\varkappa$ may be considered as its generalized inverse. Remark that G being also normal, it has a Green operator that is exactly $\varkappa$.

4.2 Representation theory, invariant subdualities

Generalities

Operator theory and representation theory are two close concepts, since one of the topic of representation theory is to represent a given group G by a subgroup of the group of linear automorphism of a given vector space. On the one hand, unitary representations are of overwhelming importance among group representations, notably for their various properties such as the Plancherel formula and their link with quantization. On the other hand, there exist topological groups with no continuous unitary representation [33]. Moreover, one sometimes restricts is attention to a given vector space (such as a subspace of the space of holomorphic functions, see [34]), and their may not exist unitary representation on these spaces (or equivalently unitary invariant spaces).

The object of this section is to show that, by using an enlarged concept of unitary operators, new unitary representations and new unitary invariant spaces may appear.

Invariant subdualities of holomorphic functions for the group of similitudes of the complex plane

Let G be a group of automorphisms acting on a set Ω (Ω *is a G-space*). The problem is to find a dual system of functions on Ω invariant under the group action, *i.e.* by defining

$$\forall g \in G, \qquad \begin{array}{rcl} \pi_g : \mathbb{C}^\Omega & \longrightarrow & \mathbb{C}^\Omega \\ f & \longmapsto & (\pi(g)f)(t) = f(g^{-1}t) \end{array}$$

find a duality such that:

$$\pi(g)(E) = E,\ \pi(g)(F) = F$$

and

$$\forall g \in G,\ (\pi(g)f, \pi(g)e)_{(F,E)} = (f,e)_{(F,E)}\ \ \forall f \in F,\, e \in E$$

In other words, calling such an operator unitary (relative to (E, F)), we look for an evaluation duality (E, F) such that the representation of the group G is unitary relatively to (E, F) *i.e.* such that each $\pi(g)$ is unitary relative to (E, F).

This problem is very general and we focus here on the particular domain $\Omega = \mathbb{C}^*$ and on the group of similitudes of the complex plane. We treat moreover two distinct problems (the first being more difficult than the second one):
- Problem 1: E and F are continuously included in the space of holomorphic functions.
- Problem 2: E and F are spaces of holomorphic functions, continuously included in $\mathbb{C}^{\mathbb{C}^*}$.

These two problems have been studied by Faraut ([15] or [14]) when G is the group of rotations of the complex plane and for Hilbert spaces. He finds that a Hilbert space H of holomorphic functions (problem 2) is invariant if and only if it is a reproducing kernel Hilbert space with the following orthonormal basis

$$\{h_m(z) = \sqrt{\mu_m} z^m,\ \mu_m \in \mathbb{R}^+,\ m \in \Lambda \subset \mathbb{Z}\}$$

with $\forall \lambda \in \mathbb{R}^*_+,\ \sum_{m \in \Lambda} \mu_m \lambda^m < \infty$ and its kernel verifies

$$K(z, w) = \sum_{m \in \Lambda} \mu_m z^m \bar{w}^m$$

Moreover, this space is continuously included in the space $\mathcal{O}(\mathbb{C}^*)$ of holomorphic functions (problem 1).

It is straightforward to see that if now the group G is the group of similitudes of the complex plane, then the reproducing kernel must be constant.

We must therefore look in an other direction, and the concept of subdualities is one.

We would like to answer completely problems 1 and 2, but we can only state following theorem:

Theorem 4.5 *If (E, F) subduality of the space of holomorphic functions (problem 1), or evaluation duality of holomorphic functions (problem 2), is invariant under G, then exists a holomorphic function $\phi : \mathbb{C}^* \longmapsto \mathbb{C}$ such that its reproducing kernel verifies*

$$K(z,w) = \phi(\frac{z}{w})$$

Conversely, to each kernel of this form is associated at least an invariant evaluation duality of holomorphic functions (problem 2).

Moreover, if the decomposition of ϕ in Laurent series is of the form

$$\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

then by decomposing $e \in E,\ f \in F$ in Laurent series:

$$e(w) = \sum_{n\in\mathbb{Z}} e_n w^n, \qquad f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$$

one has for duality product

$$(f,e)_{(F,E)} = \sum_{n\in\mathbb{Z}} \frac{f_n\, e_{-n}}{a_n}$$

Proof Let (E, F) be a subduality of the space of holomorphic functions $\mathcal{O}(\mathbb{C}^*)$ (endowed with the topology of uniform convergence on compacts). Since $\mathcal{O}(\mathbb{C}^*)$ is continuously included in the product space $\mathbb{C}^{\mathbb{C}^*}$, it follows that (E, F) is an evaluation duality. Let $K(z, w)$ be its reproducing kernel.
If the subduality is invariant under the group action, then direct calculations show that for all $g \in G$, $R(z, w) = K(g^{-1}z, g^{-1}w)$ verifies

$$R(z,.) \in E$$

$$R(.,w) \in F$$

and

$$e(w) = (R(.,w), e)_{(F,E)}$$
$$f(z) = (f, R(z,.))_{(F,E)}$$

hence R is reproducing.
By unicity of the reproducing kernel it follows that

$$\forall g \in G,\ \forall (z,w) \in {\mathbb{C}^*}^2 \quad K(z,w) = K(g^{-1}z, g^{-1}w)$$

Let now G be the group of similitudes of the complex plane, that we identify with $\mathbb{C}^*$, the group action being pointwise multiplication. Then

$$\forall (z,w) \in (\mathbb{C}^*)^2 \quad K(z,w) = K(z^{-1}z, z^{-1}w) = K(1, z^{-1}w) = \phi(z^{-1}w)$$

where ϕ is holomorphic.
The first part of the theorem is proved.
Let now K be such a kernel. Then it is straightforward to see that the primary subduality associated to this kernel is invariant and of holomorphic functions, hence the theorem is proved.

Remark 4.6 *The decomposition of ϕ in Laurent series gives a decomposition of (E, F) as a direct sum of one-dimensional invariant subdualities, and one has an analogue of a Plancherel formula. This gives the intuition that harmonic analysis on subdualities is possible.*

It is an open problem to see if, as in the Hilbertian case, any kernel of this form is associated to an invariant subduality of $\mathcal{O}(\mathbb{C}^*)$. However, if ϕ is polynomial, then the associated primary subduality is finite-dimensional hence continuously included in the space $\mathcal{O}(\mathbb{C}^*)$. In this case, the associated subduality is of course unique.

The primary subduality is however not the only possibility in case of problem 2 when ϕ is not polynomial:
let

$$E = \{e(w) = \sum_{n \in \mathbb{N}} e_n\, w^n,\ \exists \gamma \in \mathbb{R}^+,\ |n! e_n| \leq \gamma^n\ \forall n \in \mathbb{N}\}$$

and

$$F = \{f(z) = \sum_{n \in \mathbb{N}} f_{-n}\, z^{-n},\ \exists \gamma \in \mathbb{R}^+,\ |n! f_{-n}| \leq \gamma^n\ \forall n \in \mathbb{N}\}$$

We put them in duality by the following bilinear form:

$$(f, e)_{(F,E)} = \sum_{n \in \mathbb{N}} n! f_{-n}\, e_n$$

This subduality is invariant and admits for reproducing kernel function

$$K(z, w) = e^{\frac{w}{z}}$$

4.3 Berezin symbol of operators in evaluation dualities

It is well known that not all the continuous endomorphisms of $L^2(\Omega)$ are of the form

$$Tf(t) = \int_\Omega A(t, s) f(s) ds$$

In [3] D. Alpay proves that continuous endomorphisms in reproducing kernel Hilbert spaces are characterized by a function of two variables thanks to the equation

$$Tf(t) = \langle A(t, .), f(.) \rangle_H$$

and up to unitary similarity by actually a function of one single variable called the Berezin symbol (Theorem 2.4.1 p 33). This theorem extends naturally to the context of Kreĭn spaces.

In the subduality setting it appears that many morphisms in evaluation dualities are also characterized by a function of two variables:

Theorem 4.7 *Let (E, F) be an evaluation subduality on the set Ω with reproducing kernel $K(.,.)$. Then any weakly continuous operator $S : F \longrightarrow E$ and $T : E \longrightarrow E$ (resp. from E to F or from F to F) can be written as*

$$S(f)(t) = (f, \mathbf{S}(t,.))_{(F,E)}$$

$$T(e)(s) = (\mathbf{T}(.,s), e)_{(F,E)}$$

where

$$\mathbf{S}(t,s) =^t S[K(.,t)](s) = S[K(.,s)](t)$$

and

$$\mathbf{T}(t,s) =^t T[K(.,s)](t) = T[K(t,.)](s)$$

Proof For instance for S:
$S(f)(t) = (K(.,t), S(f))_{(F,E)} = (f,^t S[K(.,t)])_{(F,E)} = (f, \mathbf{S}(t,.))_{(F,E)}$

The following transposition and composition rules follow:

1. ${}^t\mathbf{S}(t,.) = S[K(.,t)] = \mathbf{S}(.,t)$, ${}^t\mathbf{T}(t,.) = T[K(t,.)] = \mathbf{T}(t,.)$
2. $T \circ S$ is associated to $[\mathbf{T} \circ \mathbf{S}](t,s) = (\mathbf{T}(.,t), \mathbf{S}(s,.))_{(F,E)}$
3. $T_1 \circ T_2$ is associated to $[\mathbf{T}_1 \circ \mathbf{T}_2](t,s) = (\mathbf{T}_1(.,s), \mathbf{T}_2(t,.))_{(F,E)}$

Remark that this generalized Berezin transform is injective and defines a non-commutative algebra of two-variable functions, the product being $\mathbf{T} * \mathbf{Q} = \mathbf{TQ}$. Moreover, in the case of holomorphic functions, it is well known that a two-variable holomorphic function is entirely defined by its restriction to the anti-diagonal $\bar{z} = w$. It follows that in case of holomorphic subdualities, and when the set Ω is conjugate symmetric, the following mapping is injective and defines a non-commutative algebra of holomorphic functions:

$$\begin{array}{rcl} B : \mathcal{L}(E,E) & \longrightarrow & \mathcal{O}(\Omega) \\ T & \longmapsto & \widetilde{T}(w) = \mathbf{T}(\bar{w}, w) \end{array}$$

the product being

$$\widetilde{T} * \widetilde{Q} = \widetilde{TQ}$$

The Berezin transform then allows one to transport operator theory problems into function theory problems using the appropriate algebra, or conversely to use operator theory arguments to solve functional problems (see for instance [36] or [30] for the use of Hilbert space operator theory to solve function theory problems).

example 1 **Polynomials, splines**

Let

$$\begin{array}{rcl} T : E_{\mathcal{P}} & \longrightarrow & E_{\mathcal{P}} \\ \sum_{i=0}^{n} \alpha_i s^i & \longmapsto & \sum_{i=0}^{n} \alpha_{(n-i)} s^i \end{array}$$

Then its Berezin symbol is given by

$$\mathbf{T}(t,s) = T[K_{\mathcal{P}}(t,.)](s) = (ts-1)^n$$

Rewriting it as

$$\mathbf{T}(t,s) = s^n(t-\frac{1}{s})^n = s^n K_{\mathcal{P}}(t,\frac{1}{s})$$

we get

$$\begin{aligned} T(e)(s) = (\mathbf{T}(.,s),e)_{(F_{\mathcal{P}},E_{\mathcal{P}})} &= \left(s^n(t-\frac{1}{s})^n, e\right)_{(F_{\mathcal{P}},E_{\mathcal{P}})} \\ &= s^n \left(K_{\mathcal{P}}(.,\frac{1}{s}), e\right)_{(F_{\mathcal{P}},E_{\mathcal{P}})} = s^n e(\frac{1}{s}) \end{aligned}$$

which gives a second expression of T.

example 2 **Entire functions and Hermite polynomials**

Let D be the differential operator:

$$\begin{array}{rcl} D : E_H & \longrightarrow & E_H \\ e & \longmapsto & e' \end{array}$$

Its Berezin symbol is

$$\mathbf{D}(z,w) = [\frac{\partial}{\partial w} K_H(z,.)](w) = 2zK_H(z,w)$$

By the transposition rule, it is also the Berezin symbol of its transpose

$${}^t\mathbf{D}(z,w) = 2zK_H(z,w)$$

and we get

$$\begin{aligned} {}^tD(f)(z) &= \left(f, {}^t\mathbf{D}(z,.)\right)_{(F_H,E_H)} = (f, 2zK_H(z,.))_{(F_H,E_H)} \\ &= 2z\,(f, K_H(z,.))_{(F_H,E_H)} = 2zf(z) \end{aligned}$$

i.e. the operator ${}^tD$ is the shift operator on the space of entire functions.

We can then recover the classical recurrence relation for Hermite polynomials:

$$\begin{aligned} H_n^{'} &= \sum_{n\in\mathbb{N}} \left(z^i, H_n^{'}\right)_{(F_H,E_H)} \frac{H_i}{i!} = \sum_{n\in\mathbb{N}} \left(z^i, DH_n\right)_{(F_H,E_H)} \frac{H_i}{i!} \\ &= \sum_{n\in\mathbb{N}} \left({}^tDz^i, H_n\right)_{(F_H,E_H)} \frac{H_i}{i!} = \sum_{n\in\mathbb{N}} \left(2z^{i+1}, H_n\right)_{(F,E)} \frac{H_i}{i!} \\ &= 2nH_{n-1} \end{aligned}$$

since $\left(z^i, H_j\right)_{(F_H,E_H)} = \delta_{i,j} i!$

example 3 **Toeplitz operator equation in holomorphic evaluation dualities** In the previous example, we have seen that the Berezin symbol of the shift operator is very simple. This is in fact true for any Toeplitz operator. Let (E,F) be a reproducing kernel duality with kernel function $K(.,.)$. If

$$\begin{array}{rcl} T_\phi : E & \longrightarrow & E \\ e & \longmapsto & \phi e \end{array}$$

the operator of multiplication by ϕ is well defined and weakly continuous then its symbol is

$$\mathbf{T}_\phi(z,w) = \phi(w)K(z,w)$$

Suppose now that E ad F are spaces of holomorphic functions, and let $\mathcal{A} = \{\widetilde{T},\ T \in \mathcal{L}(E,E)\}$ be the function algebra of the one variable Berezin symbol ($\widetilde{T}(w) = \mathbf{T}(\bar{w},w)$). The following operator equation in $\mathcal{L}(E,E)$

$$AT_\phi + BT_\psi = I$$

with ϕ and ψ given reduces to the functional equation

$$\phi(w)\widetilde{A}(w) + \psi(w)\widetilde{B}(w) = \widetilde{I}(w) = K(\bar{w},w)$$

where we look for solutions $\widetilde{A}$ and $\widetilde{B}$ in $\mathcal{A}$. This equation is very similar to a Carleson Corona problem [12].

Conclusion and comments

The concept of subduality generalizes the previous concepts of Hilbert, Kreĭn or admissible prehermitian subspaces (and also D. Alpay's concept of r.k.h.s. of pairs [2]). The set of subduality quotiented by an equivalence relation can the be endowed with the structure of a vector space isomorphic to the set of kernels and one gets a unified theory if one introduces the notions of canonical and inner subdualities.

Symplectic structure (see [27],[22]) or more generally non-symmetric structures (see for instance [18] for an example of use of non-symmetric bilinear form) are more and more used in mathematics or mathematical physics, as are non-Hermitian matrices, operators or Hamiltonians (see [16] for a good bibliography on the subject). The concept of subdualities gives a new setting to study such objects.

The link between subdualities and kernels opens new perspectives, either to study spaces or operators. The existence of the kernel may serve as a tool to study some particular dualities such as invariant dualities, or on the other hand the use of a subduality and its topological properties associated to a given kernel may help study its algebraic properties (for instance the existence of a normal subduality implies the existence of a Green operator).

Finally, we recall the statement of Laurent Schwartz concerning Hermitian subspaces [37]: "Le §12 tente une généralisation aux espaces hermitiens (à métrique non positive) et aux noyaux hermitiens associés. On rencontre là de grandes difficultés. Il apparait que (..) un noyau hermitien est associé, non plus à un sous-espace hermitien, mais à une classe de sous-espaces hermitiens; (...) Néanmoins c'est peut-être là, non pas une monstruosité, mais une nouveauté pleine d'intérêt." The generalization we propose in this article is confronted to the same difficulties. Further work is now needed to decide whether it is, as Laurent Schwartz said, a monstrosity or a novelty full of interest.

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