REVERSE ORDER LAW FOR THE GROUP INVERSE IN SEMIGROUPS AND RINGS

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Abstract

In this paper, we provide equivalent conditions for the two-sided reverse order law for the group inverse: $(ab)^{\#} = b^{\#}a^{\#}$ and $(ba)^{\#} = a^{\#}b^{\#}$, in semigroups and rings. Moreover, we prove that under finiteness conditions, these conditions are also equivalent with the one-sided reverse order law $(ab)^{\#} = b^{\#}a^{\#}$.

key words: Reverse order law; Group inverse; Green's relations; finiteness conditions; Dedekind finite ring.

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Introduction

The ordinary reverse order law (in a monoid M) states that for two invertible elements of M (elements admitting an inverse a^{-1} that satisfies $aa^{-1} = 1 = a^{-1}a$), their product is also invertible and expressed as the product of the inverses, in reverse order. That is:

(Reverse Order Law)

If a and b are invertible, then ab is invertible and $(ab)^{-1} = b^{-1}a^{-1}$.

Since the pioneering work of Von Neumann [35], various generalizations of ordinary invertibility and the ordinary inverse have been studied in semigroups and rings (with or without a unit). In this paper, we study the reverse order law for the group inverse. The group inverse admits different characterizations, one of which being as follows. Let S be a semigroup. An element a of S is group invertible if it belongs to some subgroup G_a of S, and its group inverse $a^{\#}$ is then defined as its inverse in the subgroup G_a. Such an inverse, if it exists, is unique, and is also characterized by the following three equations:

$$aa^{\#}a = a \tag{0.1}$$

$$a^{\#}aa^{\#} = a^{\#} \tag{0.2}$$

$$aa^{\#} = a^{\#}a \tag{0.3}$$

The following examples show that the reverse order law does not hold in general for the group inverse.

Example 0.1. Consider S the free semigroup generated by two idempotents e and f ($e^2 = e$ and $f^2 = f$). Then the only subgroups of S are $\{e\}$ and $\{f\}$ and ef is not group invertible, whereas e and f are.

Example 0.2. Consider $S = \mathcal{M}_2(\mathbb{R})$ and let $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $G_a = \mathbb{R}^* a$, $G_b = \{b\}$ and $G_{ab} = \{ab\}$ are subgroups of S (note that b and ab are idempotents), and $a^{\#} = \frac{1}{4}a$, $b^{\#} = b$, $(ab)^{\#} = ab = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. However $(ab)^{\#} \neq b^{\#}a^{\#}$ as $b^{\#}a^{\#} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

The purpose of the article is to give necessary and sufficient conditions under which the reverse order law for the group inverse holds locally, that is for given elements a and b. Semigroups sharing this property for all elements will then be deduced from the local statement. The study of the reverse order law for inverses traces back to the work of Greville

[17], who studied the reverse order law for the Moore-Penrose inverse of a matrix (where the Moore-Penrose inverse of a matrix A is an element $A^\dagger$ satisfying the four conditions $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$ [31]). Since then, a large amount of work has been devoted to the study of this problem, and equivalent conditions for the Moore-Penrose inverse reverse order law to hold have been proved in the setting of matrices, operators, or elements of rings with involution (see [4], [27] and references therein). In contrast, only a few studies have been conducted regarding the group inverse [5], [6], [11], [26], and the results either involve the knowledge of some group inverses ($a^\#$, $b^\#$, $(a^\# ab)^\#$, see [11] and [26], or conditions on the ring (ring of complex matrices [5], of matrices over a Bezout domain [6], or of operators [11]).

This article is divided as follows: in Section 1, we introduce the necessary definitions and notations, together with some preliminary lemmas. In Section 2, we prove equivalent conditions for the two-sided reverse order law (for the group inverse)

$$ab \text{ and } ba \text{ are group invertible and } (ab)^\# = b^\# a^\#,\ (ba)^\# = a^\# b^\#$$

in full generality for semigroups. And in Section 3, we prove that these conditions are also equivalent with the one-sided reverse order law

$$ab \text{ is group invertible and } (ab)^\# = b^\# a^\#$$

under finiteness conditions, either local (ba is group invertible) or global (S is (left) stable, R is Dedekind finite). Section 4 concludes with some general comments.

1 Notations, defintions and useful lemmas

In this paper, S is a semigroup and R is a ring with identity. All the definitions given for the semigroup S then apply to the ring R. S^1 denotes the monoid generated by S ($R^1 = R$) and for any subset $A \subseteq S$, $A' = \{x \in S,\ xa = ax\ (\forall a \in A)\}$ denotes the commutant (centralizer) of A, and A'' its bicommutant.

Let $a \in S$. We say a is (von Neumann) regular if $a \in aSa$. A particular solution to $axa = a$ is called an inner inverse, or associate, of a. A solution to $xax = x$ is called an outer (or weak) inverse. An element that satisfies $axa = a$ and $xax = x$ is called an inverse (or reflexive inverse) of a. Any regular element admits an inverse (namely the element $a' = xax$ whenever $axa = a$). Among regular elements, we will mainly be interested in completely regular elements. The element a is completely regular if there exists an inner inverse x of a commuting with a. In this case, $a' = xax$ is an inverse of a that commutes with a. A commuting inverse (or relative inverse), if it exists, is unique and denoted by $a^{\#}$. For a completely regular element a, we pose $a^0 = a^{\#}a = aa^{\#}$. The element a is group invertible if it belongs to a subgroup G_a of S. Obviously, any completely regular element belongs to the subgroup G_a of S generated by $\{a, a^{\#}\}$ hence is group invertible (and its inverse in G_a is $a^{\#}$). Conversely, if a is group invertible with inverse $a' \in G_a$, then $aa' = a'a = 1_{G_a}$ and $aa'a = 1_{G_a}a = a$, $a'aa' = 1_{G_a}a' = a'$, whence a' is the commuting inverse of S. An element is then completely regular if and only if it belongs to some subgroup of the semigroup. For this reason, completely regular elements are also called group invertible elements, or simply group elements, and $a^{\#}$ is called the group inverse of a. Finally, a is Drazin invertible if a^k is group invertible for some $k \geq 1$ [13]. The smallest of such k is called the Drazin index of a and we note $i(a) = k$ (in particular, a is group invertible if and only if its Drazin index is 1).

Next lemma will be useful in the sequel (see [2] or corollary 12 in [24] for the third point):

Lemma 1.1. *Let $a \in S$ be a completely regular element. Then*

1. *a^0 is idempotent;*

2. $a = a^2 a^\# = a^\# a^2$;

3. $a^\# \in \{a\}''$.

We say that a semigroup S is (completely) regular if all its elements are (completely) regular. S is inverse if every element $a \in S$ admits a unique inverse a'. A completely regular and inverse semigroup is called a Clifford semigroup. It is known that:

- A semigroup S is inverse if and only if it is regular and idempotents commute ([19], Theorem 5.1.1);
- In an inverse semigroup, $(ab)' = b'a'$ ([19], Proposition 5.1.2);
- A semigroup is a Clifford semigroup if and only if it is regular and its idempotents are central if and only if it is a (strong) semilattice of groups ([19], Proposition 4.2.1).

We will make use of the Green's preorders and relations in a semigroup [16]. For elements a and b of S, Green's preorders $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{H}}$ are defined by inclusions of (left, right, two-sided) principal ideals:

$$a \leq_{\mathcal{L}} b \Longleftrightarrow S^1 a \subseteq S^1 b \Longleftrightarrow \exists x \in S^1,\ a = xb;$$

$$a \leq_{\mathcal{R}} b \Longleftrightarrow aS^1 \subseteq bS^1 \Longleftrightarrow \exists x \in S^1,\ a = bx;$$

$$a \leq_{\mathcal{J}} b \Longleftrightarrow S^1 a S^1 \subseteq S^1 b S^1 \Longleftrightarrow \exists x, y \in S^1,\ a = xby;$$

$$a \leq_{\mathcal{H}} b \Longleftrightarrow (a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b)\,.$$

If $\leq_{\mathcal{K}}$ is one of these preorders, then $K_{\leq a} = \{b \in S,\ b \leq_{\mathcal{K}} a\}$ denotes the set of minorants of a, $a\mathcal{K}b \Leftrightarrow \{a \leq_{\mathcal{K}} b$ and $b \leq_{\mathcal{K}} a\}$ the equivalence relation determined by the preorder and $K_a = \{b \in S,\ b\mathcal{K}a\}$ the $\mathcal{K}$-class of a. Note that $L_{\leq a} = S^1 a$, $R_{\leq a} = aS^1$, $J_{\leq a} = S^1 a S^1$ and $H_{\leq a} = S^1 a \cap aS^1$.

We will use the following classical lemmas.

Lemma 1.2 (Cancellation). *Let S be a semigroup and $a, b \in S$. Then*

$$a \leq_{\mathcal{L}} b \Rightarrow \left(\forall x, y \in S^1,\ bx = by \Rightarrow ax = ay\right);$$

$$a \leq_{\mathcal{R}} b \Rightarrow \left(\forall x, y \in S^1,\ xb = yb \Rightarrow xa = ya\right).$$

Lemma 1.3. *Let S be a semigroup and $a, b \in S$, $c \in S^1$. Let also $\leq_{\mathcal{K}}$ be any of the preorders $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}, \leq_{\mathcal{J}}, \leq_{\mathcal{H}}$. Then*

$$ca \leq_{\mathcal{L}} a,\ ac \leq_{\mathcal{R}} a,\ aca \leq_{\mathcal{K}} a;$$

$$a \leq_{\mathcal{L}} b \Rightarrow ac \leq_{\mathcal{L}} bc \textit{ (Right congruence)};$$

$$a \leq_{\mathcal{R}} b \Rightarrow ca \leq_{\mathcal{R}} cb \textit{ (Left congruence)}.$$

We recall the following characterization of group invertibility in terms of Green's relation $\mathcal{H}$ and inverses (Theorem 2.2.5 and Theorem 2.3.4 in [19] or Corollary 3 and Corollary 4 in [25]):

Theorem 1.4. *Let a, a' be elements of a semigroup S. Then*

1. *$a^{\#}$ exists if and only if $a\mathcal{H}a^2$ if and only if H_a is a group.*
2. *Let a' be an inverse of a. Then $aa' = a'a$ if and only if $a\mathcal{H}a'$.*

Next lemma will also be useful:

Lemma 1.5. *Let $a, b \in S$, with a completely regular. Then*

$$ab = aba^{\#}a \Leftrightarrow ab \leq_{\mathcal{L}} a$$

$$b = aa^{\#}ba \Leftrightarrow ba \leq_{\mathcal{R}} a$$

Proof. Assume $ab = aba^{\#}a$. Then $ab \leq_{\mathcal{L}} a$. Conversely, assume that $ab \leq_{\mathcal{L}} a$. Then by Lemma 1.2, as $a = a(a^{\#}a)$ then $ab = aba^{\#}a$. Dually, $ba = aa^{\#}ba \Leftrightarrow ba \leq_{\mathcal{R}} a$. $\square$

2 Two-sided reverse order law for the group inverse in a semigroup

In this section, we consider the two-sided reverse order law for the group inverse, that is, for two group elements a and b:

(Two-sided Reverse Order Law for the Group Inverse)

ab and ba are group invertible and $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$.

Lemma 2.1. *Let $a, b \in S$ be group elements such that $ab \leq_{\mathcal{R}} b$ and $ba \leq_{\mathcal{L}} b$. Then $abb^\# a^\# = b^\# baa^\#$.*

Proof. By Lemma 1.5, $ab = bb^\# ab$ and $ba = bab^\# b$. Then

$$(ab)b^\# a^\# = (bb^\# ab)b^\# a^\# = b^\#(babb^\#)a^\# = b^\#(ba)a^\#.$$

□

Lemma 2.2. *Let $a, b \in S$ be group elements such that $ab \in L_{\leq a} \cap R_{\leq b}$ and $ba \in L_{\leq b} \cap R_{\leq a}$. Then ab and ba are group invertible and $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$.*

Proof. First, by Lemma 1.5, $ab = bb^\# ab = aba^\# a$ and $ba = bab^\# b = aa^\# ba$, and by Lemma 2.1, $abb^\# a^\# = b^\# baa^\#$ and $baa^\# b^\# = a^\# abb^\#$ (by symmetry).

We start by commutation:

$$(ab)b^\# a^\# = b^\#(ba)a^\# = b^\#(aa^\# ba)a^\# = b^\# a^\#(abaa^\#) = b^\# a^\# ab$$

Note that by symmetry $baa^\# b^\# = a^\# b^\# ba$. Second, we focus on inner invertibility.

$$(abb^\# a^\#)ab = (b^\# a^\# ab)ab = b^\#(a^\# aba)b = b^\# bab = ab$$

We finally adress outer invertibility. As $abb^\# a^\# = b^\# baa^\# = b^\# a^\# ab$ and $baa^\# b^\# = a^\# abb^\#$ then

$$\begin{aligned} b^\# baa^\# &= b^\# a^\# ab = (b^\# a^\# ab)b^\# b = (b^\# baa^\#)b^\# b = b^\# b(aa^\# b^\# b) \\ &= b^\# b(baa^\# b^\#) = baa^\# b^\# = a^\# abb^\# \end{aligned}$$

Finally

$$b^\# a^\# abb^\# a^\# = b^\# (a^\# abb^\#)a^\# = b^\# (b^\# baa^\#)a^\# = b^\# a^\#$$

The statement for ba follows by symmetry. □

Lemma 2.3. *Let $a, b \in S$ be group elements such that ab is group invertible and $(ab)^\# = b^\# a^\#$. Then $ab \leq_{\mathcal{H}} ba$.*

Proof. Assume $(ab)^\# = b^\# a^\#$. Then $ab\mathcal{H}b^\# a^\#$ by Theorem 1.4. As $bb^\# b^\# a^\# = b^\# a^\# = b^\# a^\# aa^\#$ then by cancellation properties (Lemma 1.2) $bb^\# ab = ab = abaa^\#$. It follows that

$$ab = bb^\# a^\# aab = b(ab)^\# abb^\# ab = bab(ab)^\# b^\# ab$$

and $ab \leq_{\mathcal{R}} ba$. Dually,

$$ab = abbb^\# a^\# a = aba^\# ab(ab)^\# a = aba^\# (ab)^\# aba$$

Finally $ab \leq_{\mathcal{H}} ba$. □

Combining these lemmas we get the following theorem:

Theorem 2.4. *Let $a, b \in S$ be group elements. Then the following statements are equivalent:*

1. *ab and ba are group invertible with $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$;*
2. *$ab\mathcal{H}ba$;*
3. *$(\exists x, y \in S^1)$ $ab = bxa$ and $ba = ayb$ ($ab \in bS^1a$ and $ba \in aS^1b$);*
4. *$ab \in L_{\leq a} \cap R_{\leq b}$ and $ba \in L_{\leq b} \cap R_{\leq a}$;*

5. $ab, ba \in H_{\leq a} \cap H_{\leq b}$*;*

6. $a^0 \in \{b\}'$ *and* $b^0 \in \{a\}'$*;*

7. $a^0, b^0 \in \{a, a^\#, a^0, b, b^\#, b^0\}'$*;*

8. The subsemigroup C *of* S *generated by* $\{a, a^\#, b, b^\#\}$ *is a Clifford semigroup.*

Proof. [(1) ⇒ (2)] By Lemma 2.3.

[(2) ⇒ (3)] Assume $ab \leq_{\mathcal{H}} ba$. Then exists $x, y \in S^1$, $ab = bax = yba$. As $bb^\# b = b$ then $ab = bax = bb^\# bax = b(b^\# yb)a$. By symmetry, $ba \in aS^1 b$.

[(3) ⇒ (4)] As $ab = bxa$ then $ab \in L_{\leq a} \cap R_{\leq b}$, and as $ba = ayb$ then $ba \in L_{\leq b} \cap R_{\leq a}$.

[(4) ⇒ (5)] Assume $ab \in L_{\leq a} \cap R_{\leq b}$. As also $ab \leq_{\mathcal{L}} b$ and $ab \leq_{\mathcal{R}} a$ by Lemma 1.3, then $ab \in H_{\leq a} \cap H_{\leq b}$. Symmetrically, $ba \in H_{\leq a} \cap H_{\leq b}$.

[(5) ⇒ (1)] Assume $ab, ba \in H_{\leq a} \cap H_{\leq b}$. Then $ab \in L_{\leq a} \cap R_{\leq b}$ and $ba \in L_{\leq b} \cap R_{\leq a}$ and by Lemma 2.2, ab and ba are group invertible with $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$.

[(1) ⇒ (6)] Assume $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$. Then $abb^\# a^\#$ is the unit of $\mathcal{H}_{ab}$, and $baa^\# b^\#$ the unit of $\mathcal{H}_{ba}$. As (1) and (1) ⇒ (2) then $ab\mathcal{H}ba$. As $ab\mathcal{H}ba$ then $abb^\# a^\# = baa^\# b^\#$ (by unicity of a unit in a group). As (2) and (2) ⇒ (3) then $ab \in R_{\leq b}$ and $ba \in L_{\leq b}$ hence $ab = bb^\# ab$ and $ba = bab^\# b$ by Lemma 1.5 and we get $(ab)(b^\# a^\#) = (bb^\# ab)(b^\# a^\#) = b^\#(bab^\# b)a^\# = b^\# baa^\#$. Symetrically, $baa^\# b^\# = a^\# abb^\#$, hence

$$abb^\# a^\# = b^\# a^\# ab = b^\# baa^\# = baa^\# b^\# = a^\# b^\# ba = a^\# abb^\#.$$

Finally, $abb^\# = (aa^\# a)bb^\# = a(a^\# abb^\#) = a(a^\# b^\# ba) = (aa^\# b^\# b)a = (b^\# baa^\#)a = b^\# b(aa^\# a) = bb^\# a$ and symetrically, $baa^\# = aa^\# b$.

[(6) $\Rightarrow$ (7)] Assume $a^0 \in \{b\}'$. As $b^\# \in \{b\}''$ by Lemma 1.1, then $b^\#$ commutes with a^0. As $b, b^\#$ commute with a^0 so does their product and $a^0 \in \{b, b^\#, b^0\}'$. As obviously $a^0 \in \{a, a^\#, a^0\}'$ then $a^0 \in \{a, a^\#, a^0, b, b^\#, b^0\}'$. Symmetrically, $b^0 \in \{b, b^\#, b^0, a, a^\#, a^0\}'$.

[(7) $\Rightarrow$ (8)] Let $c \in C$, and $c_1...c_n$ be a word representative of c in the free semigroup generated by the four elements $\{a, a^\#, b, b^\#\}$, $c = c_1...c_n$. Pose $c' = c_n^\#...c_1^\#$. As $(\forall 1 \leq k \leq n)\, c_k^\# c_k = c_k c_k^\# \in \{a^0, b^0\} \subseteq \{a, a^\#, a^0, b, b^\#, b^0\}'$, then

$$\begin{aligned}
\left(c_n^\#...c_1^\#\right)(c_1...c_n) &= \left(c_n^\#...c_2^\#\right)\left(c_1^\# c_1\right)(c_2...c_n) = \left(c_1^\# c_1\right)\left(c_n^\#...c_2^\#\right)(c_2...c_n) \\
&= \left(c_1^\# c_1\right)...\left(c_n^\# c_n\right) \text{ by induction} \\
&= \left(c_n^\# c_n\right)...\left(c_1^\# c_1\right) \text{ by commutation}
\end{aligned}$$

and by symmetry

$$(c_1...c_n)\left(c_n^\#...c_1^\#\right) = \left(c_n c_n^\#\right)...\left(c_1 c_1^\#\right) = \left(c_n^\# c_n\right)...\left(c_1^\# c_1\right)$$

Finally $c'c = c'c$ and $cc' \in C'$. It follows that

$$\begin{aligned}
\left(c_n^\#...c_1^\#\right)(c_1...c_n)\left(c_n^\#...c_1^\#\right) &= \left(c_1^\# c_1\right)...\left(c_n^\# c_n\right)\left(c_n^\#...c_1^\#\right) \\
&= \left(c_1^\# c_1\right)...\left(c_{n-1}^\# c_{n-1}\right) c_n^\# \left(c_{n-1}^\#...(c_1)^\#\right) \\
&= c_n^\# \left(c_1^\# c_1\right)...\left(c_{n-1}^\# c_{n-1}\right)\left(c_{n-1}^\#...c_1^\#\right) \\
&= \left(c_n^\#...c_1^\#\right) \text{ by induction}
\end{aligned}$$

that is $c'cc' = c'$. By the same arguments,

$$(c_1...c_n)\left(c_n^\#...c_1^\#\right)(c_1...c_n) = (c_1...c_n)$$

and c' is the group inverse of c, $c' = c^\#$. As $cc' \in C'$, then $cc^\# \in C'$. Finally, let $e \in C$ be an idempotent. Then $e^\# = e$ and $e = ee^\# \in C'$ by the previous arguments, and idempotents of C are central. As C is (completely) regular with central idempotents, it is a Clifford semigroup ([19], Proposition 4.2.1).

[(8) $\Rightarrow$ (3)] Assume C is a Clifford semigroup. As a^0, b^0 are idempotents in C then $a^0, b^0 \in C'$ and $ab = a^0abb^0 = b^0aba^0 = (bb^\#)ab(a^\#a)$ by commutation. It follows that $ab \in bS^1a$. Symmetrically $ba \in aS^1b$, and (3) is satisfied.

$\square$

Example 2.5. Let V be a vector space and $S = \mathcal{L}(V)$ be the semigroup of all linear maps $\alpha : V \to V$, acting on the left ($\alpha : x \mapsto x\alpha$). Then $(\forall \alpha, \beta \in S)\ \alpha \leq_{\mathcal{R}} \beta \Leftrightarrow N(\beta) \subseteq N(\alpha)$ and $\alpha \leq_{\mathcal{L}} \beta \Leftrightarrow R(\alpha) \subseteq R(\beta)$, where N denotes the nullspace and R the range of a linear map. Also, H_α is a group if and only if $N(\alpha) \cap R(\alpha) = \{0\}$ (see exercice 19 p.63 in [19]). Let $\alpha, \beta \in S$ be group elements. Then the two-sided reverse order law "$\alpha\beta$ and $\beta\alpha$ are group invertible with $(\alpha\beta)^\# = \beta^\#\alpha^\#$, $(\beta\alpha)^\# = \alpha^\#\beta^\#$" holds if and only if one of the following equivalent statements is satisfied:

1. $N(\alpha\beta) = N(\beta\alpha)$ and $R(\alpha\beta) = R(\beta\alpha)$;
2. $N(\beta) \subseteq N(\alpha\beta)$, $R(\alpha\beta) \subseteq R(\alpha)$, $N(\alpha) \subseteq N(\beta\alpha)$ and $R(\beta\alpha) \subseteq R(\beta)$;
3. $\pi_\alpha\beta = \beta\pi_\alpha$, $\pi_\beta\alpha = \alpha\pi_\beta$, where π_α (resp. π_β) is the projection on $R(\alpha)$ parallel to $N(\alpha)$ (resp. on $R(\beta)$ parallel to $N(\beta)$).

These conditions may be interpreted in terms of invariant subspaces: $R(\alpha), N(\alpha)$ are invariant subspaces of β and $R(\beta), N(\beta)$ are invariant subspaces of α. Also, as $R(\alpha) \oplus N(\alpha) = V$ for group elements, β and α admit a diagonal block decomposition in both $V = R(\alpha) \oplus N(\alpha)$ and $V = R(\beta) \oplus N(\beta)$.

Example 2.6. Let S be a semigroup, and $a, b \in S$ be group elements such that $ab\mathcal{H}ba$. Then by Theorem 2.4, the subsemigroup C of S generated by $\{a, a^\#, b, b^\#\}$ is a Clifford semigroup and (for instance) $(\forall n > 0)\ (aba^n)^\# = \left(a^\#\right)^n \left(b^\# a^\#\right)$.

Finally, we deduce from the "local" theorem the following "global" theorem:

Theorem 2.7. *Let S be semigroup. Then the following statements are equivalent:*

1. *S is completely regular and $(\forall a, b \in S)\,(ab)^{\#} = b^{\#}a^{\#}$;*

2. *S is regular and $(\forall a, b \in S)\,ab\mathcal{H}ba$;*

3. *S is regular and $(\forall a, b \in S)\,ab \in L_{\leq a} \cap R_{\leq b}$;*

4. *S is a Clifford semigroup.*

Proof. [(1) $\Rightarrow$ (2)] As S is completely regular, it is regular and we conclude by Lemma 2.3.

[(2) $\Rightarrow$ (3)] First, we prove that S is completely regular. Let $a \in S$. As S is regular exists $a' \in S$, $a = (aa')a = a(a'a)$. Then $a = aa'a\mathcal{H}a'a^2\mathcal{H}a^2a'$ and $a \leq_{\mathcal{H}} a^2$. As also $a^2 \leq_{\mathcal{H}} a$ by Lemma 1.3, then $a\mathcal{H}a^2$ and a is a group element by Theorem 1.4. Finally S is completely regular and we conclude by the implication (2) $\Rightarrow$ (4) in Theorem 2.4.

[(3) $\Rightarrow$ (4)] Let $a \in S$. As S is regular exists $a' \in S$, $a = aa'a, a'aa' = a'$. Then $a = (aa')a = a(a'a) \in L_{\leq aa'} \cap R_{\leq a'a}$, and it follows that $a \leq_{\mathcal{H}} a'$. Symmetrically, $a' \leq_{\mathcal{H}} a$ and finally $a\mathcal{H}a'$. By Theorem 1.4, a is completely regular. Let also $e \in S$ be idempotent. Then a and e are completely regular and by the implication (4) $\Rightarrow$ (6) in Theorem 2.4, e^0 commutes with a. But $e^{\#} = e = e^0$ hence e and a commute. It follows that idempotents are central in S, and as S is regular by assumption, it is a Clifford semigroup by Proposition 4.2.1 in [19].

[(4) $\Rightarrow$ (1)] Finally, assume that S is a Clifford semigroup. Then S is completely regular and idempotents are central and we conclude by the implication (6) $\Rightarrow$ (1) in Theorem 2.4.

$\square$

This theorem requires some comments:

- The equivalence between (1) and (4) can be interpreted in the setting of universal algebra. It then describes Clifford semigroups as the variety of unary semigroups (semigroups with a unary operation $a \to a'$, or algebras of type $(2,1)$ with a semigroup operation) satisfying four more axioms: $x = xx'x$ (regularity), $(x')' = x$ and $(xy)' = y'x'$ (involution), and $xx' = x'x$ (commutation), that is as regular unary normal involutive semigroups (the unary operation is regular, and a normal ($xx' = x'x$) involution). Though certainly known, this result appears anywhere in the previous form to the author knowledge. It can however be easily deduced from the characterization of inverse semigroups due to Schein [32] as regular unary involutive semigroups satisfying the additional equation $xx'x'x = x'xxx'$.

- A semigroup S satisfying $(\forall a, b \in S)\, ab\mathcal{H}ba$ is called a $\mathcal{H}$-commutative semigroup. By Theorem 5.1 of [29], this is equivalent with $(\forall a, b \in S)\,(\exists x, y \in S)\, ab = bxa,\ ba = ayb$. Theorem 2.7 claims that regular $\mathcal{H}$-commutative semigroups are the same as Clifford semigroups.

- The equation $(\forall a, b \in S)\, ab \in L_{\leq a} \cap R_{\leq b}$ is equivalent with $(\forall a \in S)\, aS^1 = S^1a$. As for regular semigroups $aS^1 = aS$ and $S^1a = Sa$ then assumption (3) is equivalent with S regular and normal (centric in [9]) where S is normal if $aS = Sa\,(\forall a \in S)$ [33]. Hence Theorem 2.7 claims that regular normal semigroups are the same as Clifford semigroups. The equivalence between the two notions appears for instance in [34] Theorem 2 and [22] Theorem 2 as Clifford semigroups are semilattice of groups.

3 One-sided reverse order law

Obvioulsy, Theorem 2.4 answer only partially the question: “Give equivalent conditions for the one-sided reverse order law”, for they give equivalent conditions for the two-sided reverse order law, hence only sufficient conditions for the one-sided reverse order law. In this section, we prove that under either local or global finiteness conditions, the sufficient conditions of Theorem 2.4 are also necessary.

We start with an example that shows the previous equivalent conditions are not necessary in general for the one-sided reverse order law.

Example 3.1. Consider the symmetric inverse semigroup of partial one-to-one maps on the set $X = \mathbb{R}$, $S = \mathcal{I}_{\mathbb{R}}$, where maps act on the left: $\alpha : x \mapsto x\alpha$ (S is an inverse semigroup ([19], Theorem 5.15)). Let

$$a : \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & 2x \end{array} \right. \text{ and } b : \left| \begin{array}{ccc}]-\infty,2] & \longrightarrow &]-\infty,2] \\ x & \longmapsto & \begin{cases} \frac{x}{2} & \text{if } \quad 0 \le x \le 2 \\ 1-x & \text{if } \quad -1 \le x < 0 \\ 1+x & \text{if } \quad x < -1 \end{cases} \end{array} \right. .$$

Then a and b are group elements with group inverses

$$a^{\#} : \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \frac{x}{2} \end{array} \right. \text{ and } b^{\#} : \left| \begin{array}{ccc}]-\infty,2] & \longrightarrow &]-\infty,2] \\ x & \longmapsto & \begin{cases} 2x & \text{if } \quad 0 \le x \le 1 \\ 1-x & \text{if } \quad 1 \le x < 2 \\ x-1 & \text{if } \quad x < 0 \end{cases} \end{array} \right. .$$

$a^0 = a^{\#}a$ is the identity map on $\mathbb{R}^+$ and $b^0 = bb^{\#}$ the identity map on $]-\infty, 2]$. As $ab = b^{\#}a^{\#}$ is the identity map on $[0, 1]$, ab is idempotent hence group invertible with $(ab)^{\#} = ab = b^{\#}a^{\#}$ and the reverse order law holds. However, ba maps $[-1, 2]$ onto $[0, 4]$, and is not group invertible. Also, ab and ba are not $\mathcal{H}$-related (but $ab \le_{\mathcal{H}} ba$ as expected), $a^0b \neq ba^0$ and neither of these two elements is group invertible.

Before stating the main theorems, we state a simple lemma that will be useful in the sequel.

Lemma 3.2. *Assume a, b are group invertible and $b^\# a^\#$ is a reflexive inverse of ab. Then $a^\# b^\#$ is a reflexive inverse of ba.*

Proof. First, we multiply the equation of inner invertibility $ab = abb^\# a^\# ab$ on the left by $a^\#$ and on the right by $b^\#$ to get $aa^\# bb^\# = (aa^\# bb^\#)^2$. From $b^\# a^\# = b^\# a^\# abb^\# a^\#$ we get by multiplying on the left by b and on the right by a $bb^\# aa^\# = (bb^\# aa^\#)^2$. It follows that $a^0 b^0$ and $b^0 a^0$ are idempotents. As $b^0 a^0$ is idempotent then $bb^\# aa^\# bb^\# aa^\# = bb^\# aa^\#$ and multiplying on the left by b and on the right by a we get $baa^\# b^\# ba = ba$. By symmetry $a^\# b^\# baa^\# b^\# = a^\# b^\#$. $\square$

Next theorem gives necessary and sufficient conditions for the one-sided reverse order law for ab, under the additional assumption that ba is group invertible. This may be interpreted as a "local" finiteness condition on the Drazin index: $i(ba) = 1$.

Theorem 3.3. *Let S be a semigroup and $a, b \in S$ be group elements such that ba is group invertible. Then ab is group invertible with $(ab)^\# = b^\# a^\#$ if and only if $ab\mathcal{H}ba$.*

Proof. Assume ab is group invertible with $(ab)^\# = b^\# a^\#$. Then $(ba)^2$ is group invertible with inverse $b\left((ab)^\#\right)^3 a$ (this is "Cline's formula" [10]) and

$$(ba)^2 b\left((ab)^\#\right)^3 a = b(ab)^2\left((ab)^\#\right)^3 a = b(ab)^\# a = bb^\# a^\# a = b^0 a^0$$

is the unit of the subgroup $\mathcal{H}_{(ba)^2}$. As ba is group invertible by assumption, then $ba\mathcal{H}(ba)^2\mathcal{H}b^0 a^0$ by Theorem 1.4, and $(b^0 a^0)(ba) = (ba)(b^0 a^0) = ba$. Then

$$baa^\# b^\# = (bb^\# aa^\#)baa^\# b^\# = bb^\# a^\# (abaa^\#)b^\# = bb^\# a^\# abb^\#$$

Multiplying on the left by $b^\#$ and on the right by b we get

$$b^\# baa^\# b^\# b = b^\# (bb^\# a^\# abb^\#)b = b^\# a^\# ab = abb^\# a^\# = baa^\# b^\#$$

Finally, as $ba(a^{\#}b^{\#})ba = ba$ by Lemma 3.2 then $ba = baa^{\#}b^{\#}ba = abb^{\#}a^{\#}ba$ and $ba \leq_{\mathcal{R}} ab$. Symmetrically, $a^{\#}b^{\#}ba = b^{\#}a^{\#}ab$ hence $ba = baa^{\#}b^{\#}ba = bab^{\#}a^{\#}ab$ and $ba \leq_{\mathcal{L}} ab$. Finally, $ba \leq_{\mathcal{H}} ab$ and as $ab \leq_{\mathcal{H}} ba$ by Lemma 2.3, $ab\mathcal{H}ba$. The converse is Theorem 2.4. □

Corollary 3.4. *Let S be a semigroup and $a, b \in S$ be group elements such that ab and ba are group invertible. Then $(ab)^{\#} = b^{\#}a^{\#}$ if and only if $(ba)^{\#} = a^{\#}b^{\#}$ if and only if one of the equivalent conditions of Theorem 2.4 holds.*

Example 3.5. Let S be a completely regular semigroup. Then $(ab)^{\#} = b^{\#}a^{\#}$ if and only if $ab\mathcal{H}ba$.

Example 3.6. Let R be a reduced ring (with no nilpotent elements). Then $(ab)^{\#} = b^{\#}a^{\#}$ if and only if $ab\mathcal{H}ba$. Indeed, by Cline's formula, $(ba)^2$ is group invertible hence ba is Drazin invertible. But by Theorem 5 in [13], the Drazin index is exactly 1 for Drazin invertible elements in a reduced ring. It follows that ba is group invertible, and we apply Theorem 3.3.

This example suggests the use of some "global" finiteness conditions. We start with minimal conditions on principal ideals in a semigroup (see e.g. [9]).

Definition 3.7. We say that a semigroup S satisfies the minimal condition on principal left (resp. right, resp. two-sided) ideals if every set of principal left (resp. right, resp. two-sided) ideals of S contains a minimal member with respect to inclusion, and denote this condition by $\mathcal{M}_L$ (resp. $\mathcal{M}_R$, resp. $\mathcal{M}_J$). A semigroup satisfies condition $\mathcal{M}_L^*$ (resp. $\mathcal{M}_R^*$) if any J-class of S satisfies $\mathcal{M}_L$ (resp. $\mathcal{M}_R$).

Equivalently, a semigroup S satisfies $\mathcal{M}_L$ if and only if it satifies the descending chain condition on principal left ideals (left DCCP), that is every strictly descending chain of principal left ideals of S breaks off after a finite number of terms. As remarked by Green [16], this is weaker than the minimal condition on all left ideals of S, principal and otherwise, and $\mathcal{M}_L, \mathcal{M}_R$ and $\mathcal{M}_J$ are independant in general. The definition of $\mathcal{M}_L^*$ is due to Munn [28]. It interesting to note that this condition is equivalent with the notion of left stability of Wallace and Koch [21] for the monoid S^1, where a monoid M is left stable if $a, b \in M$ and $Ma \subseteq Mab$ imply that $Ma = Mab$. Next lemma (that generalizes Theorem 8 in [16] for $\mathcal{M}_L$) is fundamental.

Lemma 3.8 (Lemma 6.41 in [9])**.** *Let S be a semigroup. Then $\mathcal{M}_L^* \Leftrightarrow \leq_{\mathcal{L}} \cap J = \mathcal{L}$.*

Example 3.9. Any finite semigroup satisfies the minimal conditions on principal left and right ideals.

Theorem 3.10. *Let S be a semigroup satisfying $\mathcal{M}_L^*$ and $a, b \in S$ be group elements. Then ab is group invertible with $(ab)^\# = b^\# a^\#$ if and only if $ab\mathcal{H}ba$.*

Proof. Assume ab is group invertible with $(ab)^\# = b^\# a^\#$. Then $ab \leq_{\mathcal{H}} ba$ by Lemma 2.3. In view of Lemma 3.8, we first prove that ab and ba are $\mathcal{J}$-related to get $ab\mathcal{L}ba$. As $ab = (ab)^2(ab)^\# = a(ba)b(ab)^\#$ then $ab \leq_{\mathcal{J}} ba$. As $ba = b^2 b^\# a^\# a^2 = b^2(ab)^\#(ab)(ab)^\# a^2$ then $ba \leq_{\mathcal{J}} ab$. Finally, $ab\mathcal{J}ba$ and by Lemma 3.8, $ab\mathcal{L}ba$. Now, as $ab \leq_{\mathcal{H}} ba$ and $ba\mathcal{L}ab$ then $ab \leq_{\mathcal{R}} b$ and $ba \leq_{\mathcal{L}} b$ and by Lemma 2.1 $abb^\# a^\# = b^\# baa^\#$.

Second, we remark that $(b^\#, a^\#)$ satisfies the same hypothesis as (a, b). The same arguments as before then give $b^\# a^\# \leq_{\mathcal{H}} a^\# b^\#$, $a^\# b^\# \mathcal{L} b^\# a^\#$ and $b^\# a^\# ab = aa^\# b^\# b$.

Third, by Lemma 3.2 $a^\# b^\#$ is a reflexive inverse of ba. Then

$$\begin{aligned} ba &= baa^\# b^\# ba = b(aa^\# b^\# b)a \\ &= b(b^\# a^\# ab)a = (b^\# ba^\# a)ba \\ &= (abb^\# a^\#)ba = ab(b^\# a^\# ba) \end{aligned}$$

and $ba \leq_{\mathcal{R}} ab$. Finally, $ba\mathcal{H}ab$.

The converse is Theorem 2.4. $\square$

Corollary 3.11. *Let S be a semigroup satisfying $\mathcal{M}_L^*$ and $a, b \in S$ be group elements. Then ab is group invertible with $(ab)^\# = b^\# a^\#$ if and only if one of the equivalent conditions of Theorem 2.4 holds.*

Example 3.12. Consider the setting of example 2.5, with V finite dimensional. Then the semigroup $S = \mathcal{L}(V)$ of endomorphisms satisfies $\mathcal{M}_L$, and the one-sided reverse-order law for the group inverse $(\alpha\beta)^\# = \beta^\# \alpha^\#$ holds for two group elements α and β if and only if $R(\alpha), N(\alpha)$ are invariant subspaces of β, and $R(\beta), N(\beta)$ are invariant subspaces of α.

Example 3.13. Let S be a right simple semigroup ($aS = S(\forall a \in S)$). Then it consists on a single $\mathcal{R}$-class (and a single J-class) and satisfies trivially $\mathcal{M}_R^*$. By Theorem 3.10 the one-sided reverse order law holds for any two group invertible elements a, b if and only if $ab\mathcal{H}ba$.

Obviously, the previous theorem can be applied to the ring case (even to non-unital rings). A ring R that satisfies the minimal condition for left principal ideals (left DCCP) is also known as a right perfect ring after a theorem of Bass ([1], Theorem **P**) who identified the two notions (where right perfect rings as defined as rings R where every left R-module has a projective cover). This is not true for semigroups, for right perfect semigroups (as studied by Isbell [20] and Fountain [15]) automatically satisfy $\mathcal{M}_L$, but the converse is not true. Theorem 3.10 then claims that in a right perfect ring R, the reverse order law $(ab)^\# = b^\# a^\#$ holds for group elements a, b in R if and only if $ab\mathcal{H}ba$. However, we will show that we can improve the theorem as follows. It is known that a right perfect ring R is Dedekind finite ([23], Proposition 6.60). Next theorem studies the reverse order law under the additional hypothesis that the ring is Dedekind-finite.

Recall that a ring R (with unit 1) is a Dedekind-finite ring if $ab = 1$ is sufficient for $ba = 1$. This is equivalent to saying invertible lower triangular matrices are exactly the matrices whose diagonal elements are invertible elements of the ring (ring "units"), and in this case the matrix inverse is again lower triangular. In particular, R is Dedekind-finite if and only if the ring $L_n(R)$ (resp. $U_n(R)$) of lower (upper) triangular matrices is Dedekind-finite (Proposition 3 in [18]).

Lemma 3.14 ([30], Proposition 4.2)**.** *If R is Dedekind-finite and the lower (upper) triangular matrice A is group invertible in $\mathcal{M}_n(R)$, then $A^\#$ is lower (upper) triangular.*

Example 3.15. A notion close to regularity is that of unit regularity. A element a of a ring R is unit regular if $a \in aR^{-1}a$, where R^{-1} is the set of "units" (invertible elements) of R. Unit regular rings are Dedekind-finite. Indeed, let $a, b \in R$ such that $ab = 1$ in a unit regular ring R. Then exists a unit $u \in R^{-1}$, $aua = a$. As $auab = au = ab = 1$, then $a = u^{-1}$ is invertible. Also $u = uab = b$ and $ba = uu^{-1} = 1$.

Theorem 3.16. *Let R be a Dedekind finite ring and $a, b \in R$ such that a, b and ab are group invertible with $(ab)^\# = b^\# a^\#$. Then ba is group invertible with $(ba)^\# = a^\# b^\#$.*

Proof. We use the following decomposition :

$$R = a^0Ra^0 \oplus a^0R(1-a^0) \oplus (1-a^0)Ra^0 \oplus (1-a^0)R(1-a^0)$$

with $a^0 = aa^\#$, and express the products in matrix form. The map $x \in R \longmapsto X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \in \mathcal{M}_2(R)$ with $x_1 = a^0xa^0, x_2 = (1-a^0)xa^0, x_3 = a^0x(1-a^0)$ and $x_4 = (1-a^0)x(1-a^0)$. Let $c = b^\#$ is then a ring homomorphism. Then we get $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}$ and $C = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}$, with $b_3 = a^0b(1-a^0) = aa^\#b - aa^\#baa^\#$ and $c_2 = (1-a^0)b^\#a^0 = b^\#aa^\# - aa^\#b^\#aa^\#$. But by lemma 2.3, $ab \leq_{\mathcal{H}} ba$ and by cancellation, as $baa^\#a = ba$ then $aba^\#a = ab$ and $b_3 = 0$, and by symmetry, $aa^\#b^\#a^\# = b^\#a^\#$ and $c_2 = 0$. Remark that c is the group inverse of b implies that C is the group inverse of

B in $\mathcal{M}_2(R)$. Now with use the Dedekind-finiteness of R. Since the matrix C is upper triangular, so is its group inverse B and $b_2 = (1 - aa^{\#})baa^{\#} = 0$. But $abaa^{\#} = ab$ hence $baa^{\#} = aa^{\#}b$. Also, since the matrix B is lower triangular, so is its group inverse C and $c_3 = aa^{\#}b^{\#}(1 - aa^{\#}) = 0$. But $aa^{\#}b^{\#}a^{\#} = b^{\#}a^{\#}$ hence $aa^{\#}b^{\#} = b^{\#}a^{\#}a$. Interchanging the role of (a, b) with $(b^{\#}, a^{\#})$ then gives $bb^{\#}a = abb^{\#}$ and condion 5) of Theorem 2.4 is satisfied, and in particular $(ba)^{\#} = a^{\#}b^{\#}$. $\square$

Corollary 3.17. *Let R be a Dedekind finite ring and $a, b \in R$ be group elements. Then ab is group invertible with $(ab)^{\#} = b^{\#}a^{\#}$ if and only if one of the equivalent conditions of Theorem 2.4 holds.*

Example 3.18. Let $(R, ., +)$ be a Dedekind-finite ring, and define $S = (R, \circ)$ the circle semigroup of R, with (Jacobson) product $a \circ b = a + b - ab\,(\forall a, b \in R)$ (this semigroup appears notably in relation with radicals, see [7], [8]). Then $\phi : (R, \circ) \to R, .)$ that maps x to $1 - x$ is a semigroup isomorphism. In particular, a is group invertible in S with group inverse $a^\flat$ if and only if $(1 - a)$ is group invertible in R, and in this case $a^\flat = 1 - (1 - a)^{\#}$. Also $a, b, a \circ b$ are group elements of $(R, \circ)$ with $(a \circ b)^\flat = b^\flat \circ a^\flat$ if and only if $(1 - a), (1 - b)$ and $(1 - a)(1 - b) = (1 - a \circ b)$ are group elements of R and $(1 - a \circ b) = (1 - a)(1 - b)$ is group invertible in R with group inverse $(1 - b)^{\#}(1 - a)^{\#}$, that is the reverse-order law holds for $(1 - a)(1 - b)$ in R. As R is Dedekind-finite this implies that $(1 - a \circ b)\mathcal{H}_R(1 - b \circ a)$ by Corollary 3.17, and the reverse-order law holds for $(1 - b)(1 - a)$. This in turns implies that $b \circ a$ is group invertible with $(b \circ a)^\flat = a^\flat \circ b^\flat$. Finally,

$$(a \circ b)^\flat = b^\flat \circ a^\flat \Leftrightarrow (1 - a \circ b)\mathcal{H}_R(1 - b \circ a) \Leftrightarrow a \circ b\mathcal{H}_S b \circ a \Leftrightarrow (b \circ a)^\flat = a^\flat \circ b^\flat.$$

Finally, next example show that in non-Dedekind finite rings, it may happen that a, b, ab are group invertible with $(ab)^{\#} = b^{\#}a^{\#}$ but ba is not group invertible.

Example 3.19. Let R be a non-Dedekind finite ring, and let $u, v \in R$ such that $uv = 1 \neq$

vu. Then $(vu)^2 = vu$. Pose $w = 1 - vu$. Then $uw = wv = 0$. The ring of 3×3 matrices over R $\mathcal{M}_3(R)$ is obviously not Dedekind finite. Consider the two following matrices of $\mathcal{M}_3(R)$

$$a = \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & w & v \end{pmatrix}.$$

Then a and b are group elements with

$$a^{\#} = \begin{pmatrix} v & w & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix}, b^{\#} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v & w \\ 0 & 0 & u \end{pmatrix}, a^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } b^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also

$$ab = \begin{pmatrix} 0 & 0 & 0 \\ 0 & vu & 0 \\ 0 & 0 & 0 \end{pmatrix} = b^{\#}a^{\#}, ba = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ w & 0 & 0 \end{pmatrix} \text{ and } a^{\#}b^{\#} = \begin{pmatrix} 0 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $ab = b^{\#}a^{\#}$ is idempotent and the reverse order law holds for ab, $(ab)^{\#} = b^{\#}a^{\#}$. However $ba(a^{\#}b^{\#}) \neq (a^{\#}b^{\#})ba$ and the reverse order law does not hold for ba. In particular, ba is not group invertible by corollary 3.4.

4 Comments

- Let M be a monoid satisfying M_L^* and assume $ab = 1$. As M is left stable and $Ma \subseteq M = Mab$ then $Ma = M$. It follows that a is also left invertible and $b = a^{-1}$, $ba = 1$. The monoid is (in some sense) "Dedekind-finite". However this condition is not sufficient for the conclusion of Theorem 3.16 to hold in general for semigroups. Indeed, consider the monoid M defined as the union of the symmetric inverse semigroup and a single element 1 acting as unit, $M = \mathcal{I}_{\mathbb{R}} \cup \{1\}$. Then $ab = 1$ implies $ba = 1$ for the only solution to $ab = 1$ is $a = b = 1$, but example 3.1 shows that the one-sided reverse order law is not equivalent with the two-sided one.

• In [3] it was proved using Core-Nilpotent Decomposition of complex matrices that: If $A, B \in \mathcal{M}_n(\mathbb{C})$ are group invertible and $AB = A^2 = BA$ then AB is group invertible with $(AB)^{\#} = B^{\#}A^{\#} = A^{\#}B^{\#}$. Theorem 2.4 proves that the additional hypothesis $AB = A^2 = BA$ is very strong since $AB = BA$ is sufficient, and the result holds not only for the ring of complex matrices but for arbitrary semigroups.

• As the ring of matrices over a Bezout domain is Dedekind finite then Corollary 3.17 generalizes Theorem 3.7 and Corollaries 3.8 and 3.9 in [6]. Indeed, any matrix A over a Bezout domain admits a representation $A = PDQ$, with $D = (\delta_{i,j})$ diagonal with either units (invertible elements) on the diagonal or 0. Pose $A' = Q^{-1}D'P^{-1}$ with D' diagonal and $\delta'_{i,i} = \delta_{i,i}^{-1}$ if $\delta_{i,j}$ is invertible, otherwise $\delta'_{i,i} = 1$. Then A' is a unit and $AA'A = A$, hence A is unit regular. As a unit regular ring is Dedekind-finite (Example 3.15), the conclusion follows.

• Theorem 3.16 raises an interesting question: is there an other characterization of the class of rings (or more generally semigroups) where the one-sided reverse order law imply the two-sided reverse order law ? The results of the paper claim that such a class contains Dedekind-finite rings and (left or right) stable semigroups. Also, Corollary 3.4 suggests the introduction of a new "group finitenness condition": $GF = \{a, b$ and ab group invertible imply ba group invertible$\}$. For semigroups with GF, the one-sided reverse order law implies the two-sided reverse order law.

• Finally, in [12], Theorem 4.8, it is claimed that, in the context of algebra of operators on a Banach space, the strong commutativity condition $aa^{\#}bb^{\#} = bb^{\#}aa^{\#}$ cannot be avoided if one wants to get the reverse order rule for the group inverse. From the previous results, we get that this is true in Dedekind-finite rings, and moreover $a^0 = aa^{\#}$ and $b^0 = bb^{\#}$ are central in C, the semigroup generated by $\{a, a^{\#}, b, b^{\#}\}$. However, this commutativity condition $a^0b^0 = b^0a^0$ is not necessary in general for semigroups, as a slight variation of example 3.1 shows (in example 3.1, the semigroup is inverse hence idempotents always commute).

Example 4.1. Consider the full transformation semigroup on the set $X = \mathbb{R}$, $S = \mathcal{T}_{\mathbb{R}}$, where maps act on the left: $\alpha : x \mapsto x\alpha$. Let

$$a : \left| \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} 2x & \text{if} \quad x \geq 0 \\ 8 & \text{if} \quad x = -1 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. \quad \text{and } b : \left| \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} 0 & \text{if} \quad x > 2 \\ \frac{x}{2} & \text{if} \quad 0 \leq x \leq 2 \\ 1-x & \text{if} \quad -1 \leq x < 0 \\ 1+x & \text{if} \quad x < -1 \end{cases} \end{array} \right. .$$

a and b are group elements with group inverses

$$a^{\#} : \left| \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} \frac{x}{2} & \text{if} \quad x \geq 0 \\ 2 & \text{if} \quad x = -1 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. \quad \text{and } b^{\#} : \left| \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} 0 & \text{if} \quad x > 2 \\ 1-x & \text{if} \quad 1 < x \leq 2 \\ 2x & \text{if} \quad 0 \leq x \leq 1 \\ x-1 & \text{if} \quad x < 0 \end{cases} \end{array} \right. .$$

Then

$$ab : \left| \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. \quad \text{and } b^{\#}a^{\#} : \left| \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. .$$

As $ab = b^{\#}a^{\#}$ is idempotent it is group invertible with $(ab)^{\#} = ab = b^{\#}a^{\#}$, and the reverse order law holds for ab. However, $(-1)a^{\#}abb^{\#} = 0$ whereas $(-1)bb^{\#}a^{\#}a = 4$ (in particular, ba is not group invertible by Corollary 3.4).

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