On generalized inverses and Green's relations

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Abstract

We study generalized inverses on semigroups by means of Green's relations. We first define the notion of inverse along an element and study its properties. Then we show that the classical generalized inverses (group inverse, Drazin inverse and Moore-Penrose inverse) belong to this class.

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There exist many specific generalized inverses in the literature, such as the group inverse ([3], [4], [5]), the Drazin inverse ([2], [1], [6]) or the Moore-Penrose inverse ([1], [6]). Necessary and sufficient conditions for the existence of such inverses are known ([3], [2], [4], [5], [7], [8], [14], [10]), as are their properties. If one looks carefully at these results, it appears that these existence criteria all involve Green's relations [3], and that all inverses have double commuting properties. So one may wonder whether we could unify these different notions of invertibility.

In this article we propose to define a new type of generalized inverse, the inverse along an element that is based on Green's relation's $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ [3] and the associated preoders. It appears that this notion encompass the classical generalized inverses but is of richer type. By deriving general existence criteria and properties of this inverse, we will then recover directly the classical results. The framework is the one of semigroups, hence the results are directly applicable in rings or algebras where generalized inverses are highly studied ([4], [5], [6], [14], [9]).

This article is organized as follows: in the first section, we review the principal definitions and theorems we will use regarding generalized inverses and Green's relations. In the second section we define our new generalized inverse, the inverse along an element, and derive its properties. In the third section we finally show that the classical generalized inverses belong to this class, and

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retrieve their properties.

1. Preliminaries

As usual, for a semigroup S, S^1 denotes the monoid generated by S. We first review some results on Green's relations, and then discuss the various notions of generalized inverses.

Green's relations

For elements a and b of S, Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are defined by

1. $a\mathcal{L}b \iff S^1a = S^1b$;
2. $a\mathcal{R}b \iff aS^1 = bS^1$;
3. $a\mathcal{H}b \iff a\mathcal{L}b$ and $a\mathcal{R}b$.

That is, a and b are $\mathcal{L}$-related ($\mathcal{R}$-related) if they generate the same left (right) principal ideal, and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. We remark that the use of the monoid S^1 instead of S allows us to rewrite relations $\mathcal{L}$ and $\mathcal{R}$ by the following equations:

$$a\mathcal{L}b \iff \exists x, y \in S^1,\ xa = b \text{ and } a = yb, \tag{1}$$

$$a\mathcal{R}b \iff \exists x, y \in S^1,\ ax = b \text{ and } a = by. \tag{2}$$

These are equivalence relations on S, and we denote the $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class) of a by $\mathcal{L}_a$ ($\mathcal{R}_a, \mathcal{H}_a$). The $\mathcal{L}$ ($\mathcal{R}$) relation is right (left) compatible, that is for any $c \in S^1$, $a\mathcal{L}b \Rightarrow ac\mathcal{L}bc$ ($a\mathcal{R}b \Rightarrow ca\mathcal{R}cb$).

Parallel with these equivalence relations we have the preorder relations:

1. $a \leq_{\mathcal{L}} b \iff S^1a \subset S^1b$;
2. $a \leq_{\mathcal{R}} b \iff aS^1 \subset bS^1$;
3. $a \leq_{\mathcal{H}} b \iff a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

These equivalence relations and preorders imply the following cancellation properties that we will frequentely use in the sequel:

$$a \leq_{\mathcal{L}} b \Rightarrow \{\forall x, y \in S^1,\ bx = by \Rightarrow ax = ay\}, \tag{3}$$

$$a \leq_{\mathcal{R}} b \Rightarrow \{\forall x, y \in S^1,\ xb = yb \Rightarrow xa = ya\}. \tag{4}$$

Finally we will also need the notion of the trace product ([11], [13]): for $a, b \in S$, we say that ab is a trace product if $ab \in \mathcal{R}_a \cap \mathcal{L}_b$.

Generalized inverses

Basically, a generalized inverse is an element that shares some (but not all) of the properties of the reciprocal inverse in a group. We review here the classical notions.

Let $a \in S$. The element a in S is called (Von Neumann) regular if $a \in aAa$, that is there exists b such that $aba = a$. In this case b is known as an inner inverse of a. If there exists $b \in S$, $bab = b$ then b is called an outer inverse (or weak inverse) of a. An element b that is both an inner and an outer inverse is usually simply called an inverse of a. If it satisfies only one of the two conditions, it is called a generalized inverse. The three most common generalized inverses (group inverse, Drazin inverse and Moore-Penrose inverse) are defined by imposing additional properties.

If b is an inverse (inner and outer) of a that commutes with a then b is a called a group inverse (or commuting inverse) of a. Such an inverse is unique and usually denoted by $a^\sharp$. Its name "group inverse" comes from the following result.

Corollary 1 (Corollary 4 p. 275 in [11]). *If a and a' are mutually inverse elements of S then $aa' = a'a$ if and only if a and a' belong to the same $\mathcal{H}$-class H. If this be the case, H is a group, and a and a' are inverses therein in the sense of group theory, i.e., $aa' = a'a = e$, where e is the identity element of H.*

This corollary itself comes from the following theorem of Green.

Theorem 2 (Theorem 7 p. 169 in [3]).

1. *If a $\mathcal{H}$-class contains an idempotent e, then it is a group with e as the identity element.*
2. *If for any $a, b \in S$, a, b and ab belong to the same $\mathcal{H}$-class H, then H is a group.*

To study non-regular elements, Drazin [2] introduced another commuting generalized inverse, which is not inner in general. An element $a \in S$ is Drazin invertible if there exists $b \in S$ and $m \in \mathbb{N}^*$ such that

1. $ab = ba$;
2. $a^m = a^{m+1}b$;
3. $b = b^2a$.

A Drazin inverse of a is unique if it exists and will be denoted by a^D in the sequel.

Finally, when S is a endowed with an involution $*$ that makes it an involutive semigroup (or $*$-semigroup), *i.e.* the involution verifies $(a^*)^* = a$ and $(ab)^* = b^*a^*$, Moore [12] and Penrose [15] studied inverses b of a with the additional property that $(ab)^* = ab$ and $(ba)^* = ba$. Once again this inverse, if it exists, is unique. It is usually called the Moore-Penrose inverse (or pseudo-inverse) of a and will be denoted by a^+.

2. A new generalized inverse: the inverse along an element

We start with a simple lemma that will give us alternative characterizations of our new generalized inverse.

Lemma 3. *Let $a, b, d \in S$. Then the two following statements are equivalent:*

1. $bad = d = dab$ *and* $b \leq_{\mathcal{H}} d$.
2. $bab = b$ *and* $b\mathcal{H}d$.

PROOF.
[1. $\Rightarrow$ 2.] Suppose $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. Then $b \leq_{\mathcal{L}} d$ and by left cancellation (equation 3) $bab = b$ and b is an outer inverse of a. But $bad = d = dab$ implies that $d \leq_{\mathcal{H}} b$ and finally $b\mathcal{H}d$.
[2. $\Rightarrow$ 1.] Naturally $b\mathcal{H}d$ implies $b \leq_{\mathcal{H}} d$. But it also implies $d \leq_{\mathcal{H}} b$ and by left and right cancellation (equations 3 and 4), $bab = b$ implies $bad = d = dab$. This ends the proof.

Definition 4. *Let $a, d \in S$. We say that $b \in S$ is* ***an inverse of* a *along* d** *if it verifies one of the two equivalent statements of lemma 3. If moreover the inverse b of a along d verifies $aba = a$, we say that b is an* ***inner*** *inverse of a along d.*

We note that ba and ab are then idempotents in the $\mathcal{R}$ and $\mathcal{L}$-class of d respectively.

Example 5. Let $S = T_3$ be the full transformation semigroup, which consists of all functions from the set $\{1, 2, 3\}$ to itself with multiplication the composition of functions. We write (abc) for the function which sends 1 to a, 2 to b, and 3 to c.
The egg-box diagram form T_3 is as follows ($\mathcal{R}$-classes are rows, $\mathcal{L}$-classes columns and $\mathcal{H}$-classes are squares; bold elements are idempotents).

(1 1 1)	**(2 2 2)**	**(3 3 3)**				
			(1 2 2), (2 1 1)	**(1 3 3)**, (3 1 1)	(2 3 3), (3 2 2)	
			(2 1 2), **(1 2 1)**	(3 1 3), (1 3 1)	**(3 2 3)**, (2 3 2)	
			(2 2 1), (1 1 2)	(3 3 1), **(1 1 3)**	(3 3 2), **(2 2 3)**	
						(1 2 3), (2 3 1), (3 1 2), (1 3 2), (3 2 1), (2 1 3)

For instance,

$$
\begin{aligned}
\mathcal{R}_{(232)} &= \{(212, (\mathbf{121}), (313), (131), (\mathbf{323}), (232)\}, \\
\mathcal{L}_{(232)} &= \{(233), (322), (\mathbf{323}), (232), (332), (\mathbf{223})\}, \\
\mathcal{H}_{(232)} &= \{(\mathbf{323}), (232)\}.
\end{aligned}
$$

Direct computations give that

- $a = (221)$ is inner invertible along $d = (232)$ with inverse $b = (323)$.
- $a' = (123)$ is invertible along $d = (232)$ with inverse $b = (323)$ but not inner invertible along d (whereas, being idempotent, it is regular).
- $a'' = (111)$ is not invertible along (232).

Of fundamental importance is the uniqueness of our new generalized inverse:

Theorem 6. *If an inverse along d exists, it is unique.*

Proof. Let b and b' be two inverses of a along d. Then $bad = d$ and $b' \leq_{\mathcal{H}} d$ implies (by right cancellation, equation 4) $bab' = b'$. But dually, $d = dab'$ and $b \leq_{\mathcal{H}} d$ implies (by left cancellation, equation 3) $b = bab'$, and finally $b = b'$.

The uniqueness of the inverse along an element allows us to introduce the following notation (on a suggestion of R. E. Hartwig): if a is invertible along d, we denote by $a^{\|d}$ the inverse of a along d.

Now we prove an interesting characterization of the inverse of a along d in terms of the group inverses $(ad)^\sharp$ and $(da)^\sharp$, as well as existence criteria.

Theorem 7. *Let $a, d \in S$. The three following statements are equivalent:*

1. *a is invertible along d.*
2. *$ad\mathcal{L}d$ and $\mathcal{H}_{ad}$ is a group.*
3. *$da\mathcal{R}d$ and $\mathcal{H}_{da}$ is a group.*

In this case

$$a^{\|d} = d(ad)^\sharp = (da)^\sharp d. \tag{5}$$

Proof.
[1. $\Rightarrow$ 2.] Suppose a is invertible along d with inverse b. Then $b\mathcal{L}d$ implies (right compatibility) that $bad\mathcal{L}dad$ and equality $bad = d$ then implies $d\mathcal{L}dad$. But $bad = d$ also implies $ad\mathcal{L}d$ and still by right compatibility, $adad\mathcal{L}dad$. Finally $(ad)^2\mathcal{L}dad\mathcal{L}d\mathcal{L}ad$.
On the other hand, $b\mathcal{R}d$ implies $adab\mathcal{R}adad$, and then equality $dab = d$ implies $ad\mathcal{R}(ad)^2$. Finally $(ad)^2\mathcal{H}ad$ and by theorem 2, $\mathcal{H}_{ad}$ is a group.

[2. $\Rightarrow$ 3.] Suppose now $ad\mathcal{L}d$ and $\mathcal{H}_{ad}$ is a group, and let (equation 1) $x \in S^1$ such that $d = xad$. From $ad = ad(ad)^\sharp ad = adad(ad)^\sharp = (ad)^\sharp adad$ we get $d = xad = xadad(ad)^\sharp = dad(ad)^\sharp$ and $da\mathcal{R}d$. To prove that $\mathcal{H}_{da}$ is a group, by theorem 2 we only need to prove that $da\mathcal{H}(da)^2$. But

$$\begin{aligned} da &= xada = xadad(ad)^\sharp a = xadad(ad)^\sharp ad(ad)^\sharp a \\ &= dadad(ad)^\sharp(ad)^\sharp a \end{aligned}$$

and $da\mathcal{R}(da)^2$. But since $d = dad(ad)^\sharp$ we have also

$$\begin{aligned} da &= dad(ad)^\sharp a = d(ad)^\sharp adad(ad)^\sharp a \\ &= d(ad)^\sharp(ad)^\sharp adada \end{aligned}$$

and $da\mathcal{L}(da)^2$. Finally $da\mathcal{H}(da)^2$.

[3. $\Rightarrow$ 1.] Suppose now $da\mathcal{R}d$ and $\mathcal{H}_{da}$ is a group. Pose $b = (da)^\sharp d$ and let (equation 2) $x \in S^1$ such that $d = dax$. Then $bada = (da)^\sharp dada = da$ implies by right cancellation $bad = d$, and $dab = da(da)^\sharp d = da(da)^\sharp dax = dax = d$. But also $b = (da)^\sharp d = (da)^\sharp dax = da(da)^\sharp x$ and $b \leq_{\mathcal{H}} d$. Finally $b = (da)^\sharp d$ is the inverse of a along d.

Example 8. Let S be the subsemigroup of $\mathcal{M}_3(\mathbb{N})$ generated by the matrices

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $a\mathcal{R}b\mathcal{R}c\mathcal{R}d$ (the semigroup is right simple), $a\mathcal{L}c$ and $b\mathcal{L}d$. There are only two $\mathcal{H}$-classes $\mathcal{H}_a = \{a, c\}$ and $\mathcal{H}_d = \{b, d\}$. In this case the egg-box diagram is very simple:

a, c	b,**d**

Since a and d are idempotents, each $\mathcal{H}$-class contains an idempotent element and hence it is a group. From theorem 7, any element is invertible along another one. Moreover, we can use equation 5 to compute the generalized inverses. For instance, the inverse of b along c is

$$b^{\parallel c} = c(bc)^\sharp = ca^\sharp = ca = c.$$

We also deduce from the previous theorem the following simple criterion for **inner** invertibility along d.

Corollary 9. *Let $a, d \in S$. Then a is* ***inner*** *invertible along d if and only if ad and da are trace products.*

PROOF.
[$\Rightarrow$] Let $b = a^{\parallel d} = d(ad)^\sharp = (da)^\sharp d$ be the inner inverse of a along d. Then by theorem 7 $ad\mathcal{L}d$ and $da\mathcal{R}d$. But by inner invertibility, $aba = a$ hence $ad(ad)^\sharp a = a = a(da)^\sharp da$ and $ad\mathcal{R}a$ and $da\mathcal{L}a$. Finally, $ad \in \mathcal{R}_a \cap \mathcal{L}_d$ and $da \in \mathcal{R}_d \cap \mathcal{L}_a$, which is the definition of trace products.
[$\Leftarrow$] Suppose now ad and da are trace products. Then $ad \in \mathcal{R}_a \cap \mathcal{L}_d$ and $da \in \mathcal{R}_d \cap \mathcal{L}_a$. By right compatibility, $ad\mathcal{L}d$ implies $ada\mathcal{L}da$. But $da\mathcal{L}a$ and once again by right compatibility, $adad\mathcal{L}ad$. On the other hand, by left compatibilty, $ad\mathcal{R}a$ implies $dad\mathcal{R}da$. But $da\mathcal{R}d$ and once again by left compatibility, $adad\mathcal{R}ad$.

By theorem 2, $\mathcal{H}_{ad}$ is a group and also $ad\mathcal{L}d$, and by theorem 7, a is invertible along d. Let $b = a^{\|d}$. Then it satisfies $bad = d$ hence $abad = ad$ and by right cancellation using $ad\mathcal{R}a$ we get $aba = a$ and the inverse is inner.

We finally prove an interesting result regarding commutativity. If A is a subset of the semigroup S, A' denotes as usual the commutant of A and A'' its bicommutant.

Theorem 10. *Let $a, d \in S$. If a is invertible along d, then $a^{\|d} \in \{a, d\}''$.*

Proof. Let b be the inverse of a along d. It then satisfies $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. Suppose $c \in \{a, d\}'$. Then

$$\begin{aligned} cd &= cbad = cdab = dacb \\ &= dc = dabc = badc = bcad \end{aligned}$$

hence $cd = cbad = bcad, \quad dc = dabc = dacb$. Then by left and right cancellation (equations 3 and 4, $b \leq_{\mathcal{H}} d$)

$$\begin{aligned} cb &= cbab = bcab \\ &= bacb = babc = bc \end{aligned}$$

and $b \in \{a, d\}''$.

We remark that if $da = ad$, the two previous results then give that $b = a^{\|d}$ commutes with a and d and that $\mathcal{H}_d = \mathcal{H}_{ad}$ is a group.

3. Inverses along d and classical inverses

Our interest in the notion of the inverse along an element is in particular based on the fact that the classical generalized inverses belong to this class.

Theorem 11. *Let $a \in S$. (S is a $*$-semigroup in 3.)*

1. *a is group invertible if and only if it is invertible along a. In this case the inverse along a is inner and coincides with the group inverse.*
2. *a is Drazin invertible if and only if it is invertible along some a^m, $m \in \mathbb{N}$, and in this case the two inverses coincide.*
3. *a is Moore-Penrose invertible if and only if it is invertible along a^*. In this case the inverse along a^* is inner and coincides with the Moore-Penrose inverse.*

In other words:

$$a^\sharp = a^{\|a}, \tag{6}$$

$$a^D = a^{\|a^m} \text{ for some integer } m, \tag{7}$$

$$a^+ = a^{\|a^*}. \tag{8}$$

Proof.

1. Suppose a is group invertible. Then by corollary 1 $\mathcal{H}_a$ is a group that contains the group inverse $a^\sharp$. Hence $a^\sharp a a^\sharp = a^\sharp$ and $a^\sharp \mathcal{H} a$, and a is invertible along a, with inverse $a^{\|a} = a^\sharp$.
   Conversely, if $a^{\|a}$ exists, then it satisfies by definition $a^{\|a}aa = a$ and $a^{\|a}aa^{\|a} = a$. But by theorem 10 it commutes with a and then satisfies also $aa^{\|a}a = a$. Hence it is the group inverse.
2. Suppose a Drazin invertible, with Drazin inverse a^D. Then by definition $a^D a a^D = a^D$ and there exists $m \in \mathbb{N}^*$, $a^D a^{m+1} = a^m = a^{m+1}a^D$. Posing $d = a^m$, we get $a^D a d = d = d a a^D$ and from $a^D a = a a^D$, $a^m (a^D)^m a^D = (a^D a)^m a^D = a^D = a^D (a a^D)^m = a^D (a^D)^m a^m$. This proves that $a^D \leq_{\mathcal{H}} d$ and finally $a^D = a^{\|a^m}$.
   Conversely, suppose there exists $m \in \mathbb{N}^*$ such that a invertible along a^m. Then $a^{\|a^m}$ is an outer inverse of a that satisfies $a^{\|a^m} a a^m = a^m = a^m a a^{\|a^m}$. But by theorem 10 it also commutes with a, hence it is the Drazin inverse.
3. Suppose a Moore-Penrose invertible with Moore-Penrose inverse a^+. Then

$$
\begin{aligned}
a^+ &= (a^+ a) a^+ = \left(a^+ a\right)^* a^+ = a^* \left(a^+\right)^* a^+ \\
a^+ &= a^+ \left(a a^+\right) = a^+ \left(a a^+\right)^* = a^+ \left(a^+\right)^* a^*
\end{aligned}
$$

   and $a^+ \leq_{\mathcal{H}} a^*$. But also

$$
\begin{aligned}
a^* &= \left(a a^+ a\right)^* = \left(a^+ a\right)^* a^* = a^+ a a^* \\
a^* &= \left(a a^+ a\right)^* = a^* \left(a a^+\right)^* = a^* a a^+
\end{aligned}
$$

   and $a^+ a a^* = a^* = a^* a a^+$. Finally a is inner invertible along a^* with $a^{\|a^*} = a^+$.
   Conversely, suppose a is invertible along a^*. Then $a^{\|a^*}$ is an outer inverse of a that satisfies

$$a^{\|a^*} a a^* = a^* = a^* a a^{\|a^*}$$

   and by involution

$$a(a^{\|a^*}a)^* = a = (aa^{\|a^*})^* a.$$

   It follows that $a^{\|a^*}a = (a^{\|a^*}a)(a^{\|a^*}a)^*$ and $aa^{\|a^*} = (aa^{\|a^*})^*(aa^{\|a^*})$ are hermitian, and $aa^{\|a^*}a = a(a^{\|a^*}a)^* = a$. Finally $a^{\|a^*}$ is the Moore-Penrose inverse of a.

Combining theorem 7 and theorem 11, we then get directly the following existence criteria and commuting relations for the classical inverses [3], [2], [4], [5], [7], [8], [14], [10].

Corollary 12. *Let $a \in S$.*

1. *A group inverse $a^\sharp$ exists if and only if $a^2 \mathcal{H} a$, in which case $a^\sharp \in \{a\}''$.*

2. *A Drazin inverse a^D exists if and only if there exists $m \in \mathbb{N}^*, a^{m+1}\mathcal{H}a^m$, in which case $a^D \in \{a\}''$.*
3. *A Moore-Penrose inverse a^+ exists if and only if $aa^*\mathcal{R}a$ and $a^*a\mathcal{L}a$, in which case $a^+ \in \{a, a^*\}''$.*

Note that many other results involving classical inverses are then straightforward consequences of theorem 11. We give two instances of this, the first one concerning equal projections (EP-elements).

In [14], we find the following proposition.

Proposition 13 (Proposition 2 p. 162 in [14]). *Given a in a ring R with involution $*$, the following conditions hold:*

1. *If $aR = a^*R$ then a^+ exists with respect to $*$ iff $a^\sharp$ exists, in which case $a^+ = a^\sharp$.*
2. *If a^+ exists with respect to $*$, $a^\sharp$ exists, and $a^+ = a^\sharp$, then $aR = a^*R$.*

Then the proof easily follows from our characterization of the group inverse and the Moore-Penrose inverse as inverses along an element:

Proof. By involution, $(aR = a^*R) \iff (Ra^* = Ra) \iff (a\mathcal{H}a^*)$. Since the inverse along an element d depends only on the $\mathcal{H}$-class of d, theorem 11 then gives the desired result.

Actually, we have the following more precise result.

Theorem 14. *Let S be a semigroup with involution $*$. Then the following statements are equivalent:*

1. *a^+ exists, $a^\sharp$ exists and $a^\sharp = a^+$.*
2. *$a\mathcal{L}a^*\mathcal{L}aa^*$.*
3. *$a\mathcal{H}aa^*$.*
4. *$a\mathcal{H}a^*a$.*
5. *$a\mathcal{R}a^*\mathcal{R}a^*a$.*

Proof.
[1. $\Rightarrow$ 2.] Suppose a^+ exists, $a^\sharp$ exists and $a^\sharp = a^+$. Then by corollary 12 $aa^*\mathcal{L}a^*$. But also $a\mathcal{L}a^\sharp\mathcal{L}a^+\mathcal{L}a^*$ and finally $a\mathcal{L}a^*\mathcal{L}aa^*$.
[2. $\Rightarrow$ 3.] By involution.
[3. $\Rightarrow$ 4.] Suppose $a\mathcal{H}aa^*$. Then by transposition $a^*\mathcal{H}aa^*$. It follows that a, a^* and aa^* are in the same $\mathcal{H}$-class, and by theorem 2 $\mathcal{H}_{aa^*}$ is a group. Then a^*a is also in the group $\mathcal{H}_{aa^*}$ and $a\mathcal{H}a^*\mathcal{H}aa^*\mathcal{H}a^*a$.
[4. $\Rightarrow$ 5.] By involution.
[5. $\Rightarrow$ 1.] Suppose $a\mathcal{R}a^*\mathcal{R}a^*a$. Then by transposition $a^*\mathcal{L}a\mathcal{L}a^*a$ and a, a^* and a^*a are in the same $\mathcal{H}$-class. By theorem 2 $\mathcal{H}_{a^*a}$ is a group. But also $a^*a\mathcal{R}a^*$ by hypothesis and by theorem 7, a is invertible along a^* or equivalently, a^+ exists. But $a\mathcal{H}a^*$ implies that $a^{\|a} = a^\sharp$ also exists and is equal to $a^{\|a^*} = a^+$.

Also, we find in [8] the following theorem.

Theorem 15 (Theorem 5.3 p. 144 in [8]). *Let R be a ring with involution $*$. An element $a \in R$ is Moore-Penrose invertible if and only if*

$$\forall x, y \in S^1,\ a^*ax = a^*ay \Rightarrow ax = ay,$$
$$\forall x, y \in S^1,\ xaa^* = yaa^* \Rightarrow xa = ya,$$

and a^*a is group invertible.
If this is the case, then also aa^* is group invertible and

$$a^+ = (a^*a)^\sharp a^* = a^*(aa^*)^\sharp.$$

Note the use of the cancellation properties. The only if part is quite strong, since we have the following result.

Theorem 16. *Let S be a semigroup with involution $*$. An element $a \in S$ is Moore-Penrose invertible if and only if*

$$\forall x, y \in S^1,\ a^*ax = a^*ay \Rightarrow ax = ay$$

and a^*a is group invertible.
If this is the case, then also aa^* is group invertible and

$$a^+ = (a^*a)^\sharp a^* = a^*(aa^*)^\sharp.$$

Proof. Suppose a^+ exists. Then a is invertible along a^*. By theorem 7, $\mathcal{H}_{aa^*}$ and $\mathcal{H}_{a^*a}$ are groups and

$$a^{\|d} = (a^*a)^\sharp a^* = a^*(aa^*)^\sharp.$$

Also $a^*a\mathcal{L}a$ and by left cancellation,

$$\forall x, y \in S^1,\ a^*ax = a^*ay \Rightarrow ax = ay.$$

Conversely, suppose a^*a is group invertible and

$$\forall x, y \in S^1,\ a^*ax = a^*ay \Rightarrow ax = ay.$$

Then by group invertibility $a^*a = a^*a(a^*a)^\sharp a^*a$ which by cancellation implies $a = a(a^*a)^\sharp a^*a$. Hence $a\mathcal{L}a^*a$, $\mathcal{H}_{a^*a}$ is a group and by theorem 7, a is invertible along a^*. But aa^* is then also group invertible. Finally, a is Moore-Penrose invertible by theorem 11 and equation 5 reads

$$a^+ = a^{\|a^*} = (a^*a)^\sharp a^* = a^*(aa^*)^\sharp.$$

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