IC RINGS AND TRANSITIVITY OF PERSPECTIVITY

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Abstract. We construct an example of an IC ring where perspectivity is transitive, but not all isomorphic idempotents are perspective. We also develop new criteria for checking perspectivity of idempotents in rings.

1. Introduction

The concept of von Neumann regularity appears ubiquitously in ring theory and semigroup theory, playing an important role in many classification theorems. We quickly review a few of the basics concepts here.

An element x of a ring R is said to be (von Neumann) *regular* if there exists some $y \in R$ such that $xyx = x$. The element y is called an *inner inverse* of x; these inverses also go by many other names in the literature. When the pair of equalities $xyx = x$ and $yxy = y$ hold, we say that x and y are *reflexive inverses* of each other. It is well known that every regular element has a reflexive inverse. A ring where every element is regular is called a (von Neumann) *regular ring*. The reader is directed to [6] for a broad overview of the history and uses of regular rings.

For a regular ring R, the following conditions are well known to be equivalent:

(1) The ring R is *unit-regular*. This means that every element of R has an inner inverse that is a unit in R. These rings were first studied by Ehrlich [3].
(2) The ring R is an *IC ring*, short for "internal cancelation ring". This means that if $R_R = A \oplus B = A' \oplus B'$ and $A \cong A'$, then $B \cong B'$. In other words, isomorphic summands of R_R have isomorphic complements. The reader is directed to [9] for additional information on these rings.
(3) The ring R is a *perspective ring*. This means that isomorphic summands of R_R have a common complement. Summands with a common complement are said to be *perspective* as well. Perspective rings were first studied generally in [5].
(4) The ring $\mathbb{M}_2(R)$ has *transitivity of perspectivity*. In other words, if A, B, C are three summands of $\mathbb{M}_2(R)_{\mathbb{M}_2(R)}$, with A perspective to B, and B perspective to C, then A is perspective to C. (Conveniently, one can, instead, work with the conceptually easier module R^2_R, rather than $\mathbb{M}_2(R)_{\mathbb{M}_2(R)}$.)
(5) The ring R has *stable range one*. This means that for any $c, d \in R$ with $cR + dR = R$, then there exists some $z \in R$ with $c + dz$ a unit in R. The stable range one condition was first introduced by Bass [1].

The equivalence of the first four conditions was originally proved by Handelman (see Theorem 2 and Theorem 15 in [8]), and the fifth condition was added due to the independent work of Fuchs [4, Corollary 1 and Theorem 4] and unpublished work of Kaplansky.

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For a general ring R, we have the following chart of implications:

$$\begin{tikzcd}
\mathbb{M}_2(R) \text{ has transitive perspectivity} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & R \text{ has stable range one} \\
R \text{ is perspective} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
R \text{ has transitive perspectivity} & R \text{ is an IC ring.}
\end{tikzcd}$$

The interested reader is directed to [10] for the history regarding the proofs of these implications, as well as examples showing that there are no other implications among these properties (besides those obtained by following consecutive arrows).

While an IC ring is not necessarily regular, let alone unit-regular, it comes very close. Being an IC ring is equivalent to asserting that every regular element has a unit inner inverse (and the proof is implicit in Ehrlich's original work in [3]; see [9]). Thus, this raises the tantalizing possibility that any IC ring with transitive perspectivity must be perspective. The main result of our paper is the construction of a counterexample to that prospect.

Theorem 1.1. *There exists an IC ring with transitive perspectivity that is not a perspective ring.*

We construct our example in Section 3. Before that, in Section 2, we develop new, simple criteria for checking perspectivity in rings, which may be of independent interest.

All rings in this paper are unital and associative, but not necessarily commutative.

2. Perspectivity and idempotents

Given a ring R, we let $\mathrm{idem}(R)$ denote its set of idempotents, and we let $\mathrm{U}(R)$ denote its set of units. We will also adapt these notations to semigroups S. We assume that the reader is familiar with the basic facts about idempotents in [11, Section 21], especially regarding isomorphic, conjugate, and left (or right) associate idempotents.

Given $e, f \in \mathrm{idem}(R)$, we write $e \sim_\ell f$ to denote that e and f are left associates; i.e., $Re = Rf$. Similarly, we write $e \sim_r f$ when e and f are right associates. Those familiar with semigroup theory will recognize that $e \sim_\ell f$ is equivalent to saying that e and f are related by Green's $\mathscr{L}$ relation. The reason for our distinct notation will become apparent shortly.

More generally, given $e, f \in \mathrm{idem}(R)$, we say that e is connected by a *left n-chain* to f if there exist idempotents $g_0, g_1, \ldots, g_n \in \mathrm{idem}(R)$ such that

$$e = g_0 \sim_\ell g_1 \sim_r g_2 \sim_\ell g_3 \sim_r \cdots g_n = f.$$

Right n-chains are defined similarly. When we write $e \sim_{\ell r} f$, we will mean that e is left 2-chained to f, and similar notation will be used for left and right n-chains between idempotents.

It is well known that $e \sim_\ell f$ is equivalent (even in semigroups) to the pair of equalities $ef = e$ and $fe = f$. Over rings more is true; the left associates of an idempotent e are in bijective correspondence with the conjugates $u^{-1}eu$ where $u \in 1 + (1-e)Re \subseteq \mathrm{U}(R)$. Thus, the set $(1-e)Re$ parameterizes the left associates of e. Writing $u = 1 + x$ with $x \in (1-e)Re$, then $u^{-1} = 1 - x$. We thus have that $e \sim_\ell f$ holds if and only if there exists some $x \in (1-e)Re$ such that $f = (1-x)e(1+x)$. (By replacing x with $-x$, one also sees that the placement of the $+$ and $-$ symbols in that last equality can be reversed, if desired.)

We will make use of the following simple fact.

Lemma 2.1. *Let S be a semigroup, let $g, h \in \mathrm{idem}(S)$, and let $u \in \mathrm{U}(S)$. We have $g \sim_\ell h$ if and only if $u^{-1}gu \sim_\ell u^{-1}hu$.*

Proof. Assume $g \sim_\ell h$. Thus, we know that $gh = g$ and $hg = h$. We then have

$$(u^{-1}gu)(u^{-1}hu) = u^{-1}gu$$

and

$$(u^{-1}hu)(u^{-1}gu) = u^{-1}hu,$$

or equivalently, $u^{-1}gu \sim_\ell u^{-1}hu$. The converse follows similarly. $\square$

This argument also explains why idempotents in rings that are left or right n-chained must be conjugate, and in particular are isomorphic. Recall that two idempotents $e, f \in \mathrm{idem}(S)$ are isomorphic if and only if there exist reflexive inverses $a, b \in R$ such that $e = ab$ and $f = ba$. The following fact about isomorphic 2-chained idempotents is found implicitly in [7] and [12], but does not seem well known. We include a short, direct proof.

Proposition 2.2. *Let S be a semigroup, and let $a, b \in S$ be reflexive inverses. Setting $e = ab$ and $f = ba$, then $e \sim_{\ell r} f$ if and only if $ebe \in \mathrm{U}(eSe)$, or equivalently $fbf \in \mathrm{U}(fSf)$. Similarly, $e \sim_{r\ell} f$ if and only if $eae \in \mathrm{U}(eSe)$, or equivalently $faf \in \mathrm{U}(fSf)$.*

Proof. We prove only the first equivalence, as the others follow by symmetry considerations.

($\Rightarrow$): Assume $e \sim_\ell g \sim_r f$ for some $g \in \mathrm{idem}(S)$. Thus $eg = e$, $ge = g$, $fg = g$, and $gf = f$.

We claim that ebe and $eage$ are inverses in eSe. Indeed, we compute

$$(ebe)(eage) = ebea(ge) = e(baba)g = efg = eg = e$$

and similarly

$$(eage)(ebe) = ea(ge)(be) = eag(bab) = ea(gf)b = eafb = eabab = e^3 = e.$$

($\Leftarrow$): Assume $x \in eSe$ is an inverse to ebe. Thus,

$$e = (ebe)x = ebx \tag{2.3}$$

as well as

$$e = x(ebe) = (xe)(be) = x(bab) = xb. \tag{2.4}$$

Set $g = bx$. Using (2.4) we compute

$$g^2 = bxbx = bex = bx = g$$

so $g \in \mathrm{idem}(S)$. Also

$$ge = bxe = bx = g,$$

while (2.3) says $eg = e$. Hence $e \sim_\ell g$.

Finally,

$$fg = babx = bx = g$$

and, by (2.4),

$$gf = (bx)(ba) = b(xb)a = b(ab)a = f^2 = f.$$

Thus $g \sim_r f$, and therefore $e \sim_{\ell r} f$. $\square$

We can leverage Proposition 2.2 to obtain a simple criterion for 3-chaining of isomorphic idempotents.

Theorem 2.5. *Let R be a ring, and let $a, b \in R$ be a pair of reflexive inverses. Setting $e = ab$ and $f = ba$, then $e \sim_{\ell r\ell} f$ if and only if there exists some $x \in (1-e)Re$ such that $ea(1+x)e \in \mathrm{U}(eRe)$. (Symmetrically, $e \sim_{r\ell r} f$ if and only if there exists some $y \in eR(1-e)$ such that $e(1+y)be \in U(eRe)$.)*

Proof. Assume $e \sim_{\ell r\ell} f$. This is equivalent to the existence of some $x \in (1-e)Re$ such that $(1-x)e(1+x) \sim_{r\ell} f$. In other words, after conjugating, $e \sim_{r\ell} (1+x)f(1-x)$.

Note that $(1+x)f(1-x) = [(1+x)b][a(1-x)]$ and $e = ab = [a(1-x)][(1+x)b]$. So by Proposition 2.2, this is equivalent to asserting $ea(1-x)e \in U(eRe)$. After replacing x by $-x$, we are done. □

These ideas can be inductively continued, thus providing (slightly more complicated) criteria for n-chaining of isomorphic idempotents, for any $n > 3$.

3. The Main Example

Let D be the subset of $\mathbb{Z} - \{0\}$ consisting of those integers whose prime factors are all congruent to $\pm 1 \pmod 8$. Note that D is a multiplicatively closed subset of $\mathbb{Z}$, and fix $T = D^{-1}\mathbb{Z}$, which is a subring of $\mathbb{Q}$. It makes sense to talk about congruence modulo 8 in T; also note that any element of T that is not congruent to $\pm 1 \pmod 8$ is not a unit. We let ν denote the 2-adic valuation on $\mathbb{Q}$, which is also defined on T.

Fix

$$R = \begin{pmatrix} T & 4T \\ 4T & T \end{pmatrix},$$

which is a subring of $\mathbb{M}_2(T)$. Our ultimate goal will be to prove that R is an IC ring, with transitive perspectivity, and yet show that it is not a perspective ring.

We begin by classifying the nontrivial idempotents of R, so let $E \in \mathrm{idem}(R) - \{0, I\}$. Since T is a commutative domain, then in the bigger ring $\mathbb{M}_2(T)$ we have that every nontrivial idempotent is isomorphic to $E_{1,1}$. Thus, we may write $E = BA$ and $E_{1,1} = AB$, for some reflexive inverses $A, B \in \mathbb{M}_2(T)$. Now $A = ABA = E_{1,1}A$, and similarly $B = BE_{1,1}$, so we can write

$$A = \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} t & 0 \\ u & 0 \end{pmatrix},$$

for some $r, s, t, u \in T$. Since $AB = E_{1,1}$, we have $rt + su = 1$. Thus, either $\nu(rt) > 0$ and $\nu(su) = 0$, or vice versa. Also, as $BA \in R$ we must have $\nu(ru) \geq 2$ and $\nu(st) \geq 2$. Thus, the only possibilities are either $\nu(r) \geq 2$ and $\nu(t) \geq 2$, or $\nu(s) \geq 2$ and $\nu(u) \geq 2$. In the latter case, $A, B \in R$ and so E is isomorphic to $E_{1,1}$ (over R). In the first case, after switching the rows of A and the columns of B, we see that E is isomorphic to $E_{2,2}$ (over R).

On the other hand, the idempotents $\{0, I, E_{1,1}, E_{2,2}\}$ are not isomorphic over R, since their images are not isomorphic in the factor ring $R/2R \cong \mathbb{F}_2 \times \mathbb{F}_2$. Thus, there are exactly four equivalence classes of isomorphic idempotents in R.

Further, notice that if E is isomorphic to $E_{1,1}$, then $I - E$ is nontrivial, and cannot be isomorphic to $E_{1,1}$ (by the same argument as in the previous paragraph), and thus $I - E$ is isomorphic to $E_{2,2}$. Similarly, if $E \cong E_{2,2}$ then $I - E \cong E_{1,1}$. This shows that for nontrivial isomorphic idempotents, their complements are also isomorphic over R. The same holds true of the trivial idempotents. Thus, R is an IC ring by [9, (1.4)].

Next, putting

$$A = \begin{pmatrix} 5 & 8 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 13 & 0 \\ -8 & 0 \end{pmatrix},$$

then a quick check shows that A and B are a pair of reflexive inverses. We claim that $AB = E_{1,1}$ is not left 3-chained to BA. To see this, we apply Theorem 2.5. The criterion reduces to showing that $5+8x$ is not a unit in T, for any $x \in 4T$, which is clear since all units are congruent to $\pm 1 \pmod 8$. In any perspective ring, all pairs of isomorphic idempotents are left 3-chained according to [2, Lemma 6.3] (and its dual). This shows that R is not perspective.

Finally, we show that perspectivity is transitive in R. It suffices, by [2, Section 6], to show that if $E \sim_{\ell r \ell} F$, then $E \sim_{r \ell r} F$. When E is a trivial idempotent, then $F = E$, and the condition holds. So, by symmetry, and after conjugating if necessary, it suffices to consider the case when $E = E_{1,1}$. Suppose $E_{1,1} \sim_{\ell r \ell} F$. Writing $E_{1,1} = AB$ and $F = BA$, for some reflexive inverses $A, B \in R$, then repeating the computation done at the beginning of this sections (with minor modifications) shows that we can write

$$A = \begin{pmatrix} r & 4s \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} t & 0 \\ 4u & 0 \end{pmatrix},$$

for some $r, s, t, u \in T$ with $rt + 16su = 1$. Since $E \sim_{\ell r \ell} F$, then the criterion in Theorem 2.5 gives some element $x \in 4T$ such that $r + 4sx \in \mathrm{U}(T)$. This is possible only when $r \equiv \pm 1 \pmod 8$. From $rt + 16su = 1$, we then know $t \equiv \pm 1 \pmod 8$.

Write $t = t_1/t_2$ and $u = u_1/u_2$ with $t_1, u_1 \in \mathbb{Z}$ and $t_2, u_2 \in D$. Note, in particular, that $t_1 u_2 \equiv \pm 1 \pmod 8$. By Dirichlet's theorem on primes in arithmetic progressions, we can fix some $y \in 4\mathbb{Z}$ such that $t_1 u_2 + 4 t_2 u_1 y \equiv \pm 1 \pmod 8$ is prime in $\mathbb{Z}$. Thus $t + 4uy \in \mathrm{U}(T)$, and so by the (dual version of) Theorem 2.5 we have $E \sim_{r \ell r} F$, as desired.

Remark 3.1. (1) One might wonder why D, the set of denominators, was chosen as above. First, the proof that R is not perspective, given above, uses the fact that any element congruent to 5 modulo 8 is not a unit in T. While the exact congruence class could have been changed, the underlying idea requires that D avoid infinitely many primes. On the other hand, our use of Dirichlet's theorem requires D to contain infinitely many primes.

This would seem to open up the possibility of inverting all the primes congruent to 1 modulo 4, and not those which are 3 modulo 4. However, since $-1 \in \mathrm{U}(\mathbb{Z})$, primes are really only defined up to sign. Rather than work with only *positive* primes, it is much more convenient to work with general congruence classes.

(2) Another question one might ask is whether we can replace R by the simpler ring

$$R' = \begin{pmatrix} T & 2T \\ 2T & T \end{pmatrix}.$$

The proof that R' is an IC ring that not a perspective ring is exactly the same as for R. However, perspectivity is not transitive. Indeed, taking

$$A = \begin{pmatrix} 5 & -2 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 0 \\ 12 & 0 \end{pmatrix},$$

then a quick computation shows that A and B are reflexive inverses and $AB = E_{1,1}$. Since $5 + (-2)2 = 1 \in \mathrm{U}(T)$, we have that AB is connected by a left 3-chain to BA. Yet, for any $y \in 2T$ we see that $5 + 12y \equiv 5 \pmod 8$, and hence $5 + 12y$ is never a unit in T. Thus, AB is not connected by a right 3-chain to BA.

4. Another consequence

The methods used in this paper also give us an alternative proof of the following recent result:

Proposition 4.1. *Let R be a ring. If $\mathbb{M}_2(R)$ has perspectivity transitive, then R has stable range one.*

Proof. Assume $cr + ds = 1$ for some $c, d, r, s \in R$. Taking

$$A = \begin{pmatrix} c & ds \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} r & 0 \\ 1 & 0 \end{pmatrix},$$

then $AB = E_{1,1}$. Thus, one can quickly check that A and B are reflexive inverses. Also, taking $x = 1-r \in R$, then $r+1\cdot x \in \mathrm{U}(R)$. Thus, by Theorem 2.5, we have $AB \sim_{r\ell r} BA$. By transitivity of perspectivity, $AB \sim_{\ell r\ell} BA$. Equivalently, $c + (ds)y \in \mathrm{U}(R)$ for some $y \in R$, which is what was needed to verify that R has stable range one. □

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