Generalized inverses, semigroups and rings

- Mémoire d'habilitation à diriger des recherches -
(Professorial thesis)

Xavier Mary

Laboratoire Modal'X, Université Paris-Nanterre

Foreword

What is (and what is not) this memoir.

- This memoir is a professorial thesis in mathematics. As such, it presents the results obtained in my mathematical career after my PhD thesis (all of them either published or in the submission/referee process). But this memoir does not only contain my mathematical achievements. It also tries to display my personal eye on the research in mathematics. And, of course, it contains lots of notations, definitions, equations and theorems!
- Despite this, I tried to make this memoir not just a list of theorems. Indeed, while part of my research is of the "problem solving" type, some contributions are more of the "building" type (building tools, building bridges between theories, ...). Therefore, while it states some theorems, it also introduces many definitions, and presents personal comments on my own understanding of the notions and problems at stake. Also, while it is largely devoted to my personal achievements (with or without co-authors), it also contains either seminal results I used thoroughly in my research, or recent progress on a close topic by others mathematicians. When citing such results, I will add a * after the reference\ Theorem[1].
- Finally, this memoir tries not to be a simple enumeration of articles and the results therein. While some sections indeed refer to a single article, I also tried to combine the results of different articles and take a step back whenever it was possible, to propose a more global, accurate and up-to-date presentation of certain notions. I also chose to leave aside certain results, generally because they were too technical and specific to be presented shortly and simply. However, all main results are presented.

Which themes are discussed?

While I worked during my PhD thesis at the interplay between statistical learning theory and functional analysis, my research since then is very far away from this starting point. Indeed, it now takes place in the field of **non-commutative algebra**, and

[1] For instance: [166, Theorem 3]*

deals both with properties of **elements in algebraic structures** such as semigroups and rings (but also unary semigroups, S-acts and modules), and properties of **these algebraic structures themselves**.

Within this very general field, there are several guidelines that inspired my research. First come generalized inverses, that inspired me in the first place and enlighten the path to more general semigroup theory and ring theory. Then among these generalized inverses, and apart the new inverse along an element that I introduced in [146], I study particularly the special case of group inverses, which are commuting reflexive inverses, but can also be seen as genuine inverses of units in a local submonoid eSe, where $e \in E(S)$ is an idempotent, inner inverses along an idempotent or "idempotents modulo Green's relation $\mathcal{H}$". Naturally, (genuine) idempotents play an important part of my studies, as well as the relation of association between them and the natural partial order on them (and more generally, the natural partial order on arbitrary elements). And finally, most of my research uses Green's relations and their extensions. But these are rather general subjects, on none of these alone subsumes my research.

If I had to single out a few themes that embraces (nearly) all my work, one would be Miller and Clifford's trace product theorem ([166, Theorem 3]*, Theorem 1.1.1 in the present memoir). It serves as a basis for many results and many theorems (some about generalized inverses, some on pure semigroup theory and others on chains of idempotents in rings) can be derived from an extension of their theorem to products of the form $azb \in R_a \cap L_b$ (see Section 5.2 and in particular Theorem 5.2.1 therein). Another one would be Green's relation $\mathcal{H}$, with a particular emphasis on the "equation" $dad\,\mathcal{H}\,d$ (a special case of the above equation, since $R_d \cap L_d = H_d$ by definition). Indeed, this equation characterizes the existence of the inverse of a along d [146], but also, taking $a = 1$ (or d), we get that d is group invertible and, finally, for d fixed, the equation $dad\,\mathcal{H}\,d$ corresponds to a being an inner inverse of d modulo $\mathcal{H}$ [158]. I also studied semigroups where $\mathcal{H}$ is a congruence (cryptic semigroups) in [148], or $\mathcal{H}$-commutation properties in [148] and [149]. And extensions of Green's relations (notably $\mathcal{H}$) were used in [150]. Another theme is the study of semigroups (or semigroup biacts) whose structure resembles that of completely simple, completely regular or inverse semigroup ([148], [150], [152]). And a last one the interplay between direct summands of modules and idempotents and generalized inverses in the monoid part of the endomorphism ring of that module [117], [141], [156].

At a more conceptual level, I have been searching for **connections between element-wise and global properties** of semigroups and rings, and even more fundamentally, I have been looking for some generality behind specificities.

Hopefully did I managed to reach some of these goals in my work, as this memoir will try to show.

Which themes are **not** discussed?

As us usual for a professorial thesis, this memoir does not discuss the results obtained during my PhD thesis. Neither does it present some more recent results obtained at the interplay between statistics, probability and linear algebra in my study of *determinantal* *sampling designs* [9], [31], [135]. While of independent interest, they clearly do not belong to same field as the other articles discussed here.

How is the memoir divided?

The memoir is divided into five parts, each one (except the first -introductory- one) focusing on a a particular domain of my research, and a specific area of mathematics. To summarize, Part II and Part III deal with generalized inverses, Part IV with the structure of semigroups and Part V with ring and module theory. I chose to make these four last parts self-contained, in order to be readable on their own by those specialists of a specific domain, and thus without any knowledge of the other parts. This inescapably implies some repetitions (for those brave enough to read the memoir entirely!) since these domains are yet interrelated.

Part I is dedicated first to the necessary definitions and notations. It presents notably Green's relations, regularity, generalized inverses, and perspectivity of direct summands of a module. In a second time, it presents the (successful and well-known) theory of group invertible elements, completely regular semigroups and strongly regular rings. This is the occasion to investigate on an accomplished theory the connections between the different notions addressed in this memoir, and more generally in my research. Also, this very beautiful and complete theory can be seen in some way as the mathematical guidance for my research in semigroup and ring theory by using generalized inverses and/or Green's relations.

In Part II I present the notion of *inverse along an element* that was introduced in 2011 [146], after my two first encounters with generalized inverses [144], [145]. This is certainly my major contribution to the field of generalized inverses, for it led to many subsequent work (7 of my articles deal directly with this notion, but, more generally, one can now find more than 70 articles citing [146], according to MathSciNet). General results as well as inverses along specific elements are studied both in the semigroup and ring context. Other questions widely studied in the generalized inverse community - such as reverse order laws, Cline's formula or Jacobson lemma - are also studied for this inverse along an element (notably in case this element is a commuting or bicommuting idempotent).

In Part III, I gather my results that deal specifically with the group inverse. The group inverse actually shows up in most of my articles, and the first chapter of the part (Chapter 8) gathers various existence criteria obtained in the course of my research. In the three next chapters I focus mainly on three papers. The first one [159] studies the group inverse of a product ab in a ring, where the elements a and b are merely assumed regular (Chapter 9). The second one [149], whose results are presented in Chapter 10, also studies the group inverse of a product, but in another perspective. It combines semigroup and ring theory, and answers the question of the reverse order law for the group inverse. The third paper [160] is of different flavor. It compares different extensions of unit-regularity in non-unital rings, some of which based on the group inverse. Such extensions are discussed in Chapter 11. Finally, in the last chapter of the part (Chapter 12), connections between group inverses and special clean elements of a ring are also discussed.

Part IV exposes my contribution to the theory of the algebraic structure of semigroups, notably through the presentation of four articles [79], [148], [150], [152]. One common feature of these works is the major use of Green's relations (another one is that all four articles have been published in Semigroup Forum!). Two chapters of Part IV are not of this kind. Chapter 13 presents the main results regarding the inverse along an element from a semigroup point of view, while Chapter 16 gathers some semigroup results originally obtained in the study of perspective modules and rings.

Finally, Part V presents some recent module and ring theoretical results ([117], [139], [140], [141], [151], [153], [156], [160], [161]), that share in common the use of generalized inverses to study apparently purely additive notions such as clean elements (in rings) and perspectivity of direct summands (in modules). In this part, and thanks to the contribution of D. Khurana and P.P. Nielsen, we will also encounter some results in number theory!

As this memoir is dedicated to a audience of mathematicians in various fields and of different interest, I suggest the following choices of lectures depending on the interest of the reader (apart Part I, dedicated to all of them):

- For those specialist of *generalized inverses*, I suggest the reading of Parts II and III (in particular Chapters 8, 9 and 10 in Part III);
- For researchers in *semigroup theory*, I suggest to read Part IV. Chapters 8, 10 and 11 (in Part III) may also be of interest.
- Finally, for those working interested in *modules and rings*, I suggest to read primarily Part V. In a second time, they might also find some interest in Chapter 9, Chapter 11 (in particular Section 11.4) and Chapter 12 within Part III.

Special thanks

I wish to express my deepest gratitude to all my co-authors, from whom I learned a lot on many subjects, from matrix theory to Morita context or number theory. I also thank them for the very stimulating conversations we had during our collaboration, and their kindness in all domains. Besides, I express my gratitude to P. Patricio, V. Gould and A. Leroy (in order of appearance in my scientific life) for introducing me into their respective community; researchers in the field of generalized inverses, semigroup theorists and ring theorists respectively. This is especially important for me in regards of the lesser visibility of these themes in the French mathematical community than in some other countries.

V. Gould and A. Leroy, rejoined by D. Mosic, also kindly accepted the ungrateful task of reporting this memoir. Let me again gratefully thank them!

Finally, I thank my beloved wife and children for their patience and support not only during the redaction of this memoir, but also during all these times when difficult (at least for me) mathematical issues kept me busy and away from them probably more than it should have been.

Contents

List of publications

Ref.	Year	Co-author(s)	Title	Journal	Subject(s)	Chapter(s)
[144]	2008		Moore-Penrose Inverse in Kreĭn Spaces	Integral Equ. Oper. Theory	GI, Green's Rel.	3, 7
[145]	2008		On the converse of a theorem of Harte and Mbekhta: Erratum to "On generalized inverses in C^*-algebras"	Studia Mathematica	GI, Green's Rel.	2, 3
[146]	2011		On generalized inverses and Green's relations	Linear Algebra Appl.	GI, Green's Rel.	3, 4, 5, 13, 14
[147]	2012		Natural Generalized Inverse and Core of an Element in Semigroups, Rings and Banach and Operator Algebras	Eur. J. Pure Appl. Math.	GI, Green's Rel.	3, 4, 7, 13, 21, 22
[157]	2012	P. Patrício	The inverse along a lower triangular matrix	Applied Math. Comp.	GI, Green's Rel., Ring El.	3, 7
[158]	2013	P. Patrício	Generalized inverses modulo $\mathcal{H}$ in semigroups and rings	Linear Multilinear Algebra	GI, Green's Rel., Ring El.	3, 7, 13, 14
[148]	2014		Classes of semigroups modulo Green's relation $\mathcal{H}$	Semigroup Forum	GI, Green's Rel., Struct. Sem.	3, 13, 14
[149]	2015		Reverse order law for the group inverse in semigroups and rings	Comm. Algebra	Group inverse, Green's Rel.	10, 21
[159]	2016	P. Patrício	The group inverse of a product	Linear Multilinear Algebra	Group inverse, Green's Rel., Ring El.	9
[150]	2017		On $(E, \widetilde{H}_E)$-abundant semigroups and their subclasses	Semigroup Forum	Green's Rel., Struct. Sem.	10, 15
[151]	2017		Weak inverses of products – Cline's formula meets Jacobson lemma	J. Algebra Appl.	GI, Green's Rel., Ring El.	3, 4, 7, 13, 21, 22
[225]	2017	H. Zhu and J. Chen and P. Patrício	Centralizer's applications to the inverse along an element	Applied Math. Comp.	GI, Green's Rel.	3, 4, 7
[152]	2018		A local structure theorem for stable, $\mathcal{J}$-simple semigroup biacts	Semigroup Forum	Green's Rel., Struct. Sem.	17
[78]	2018	A. Guterman and P. Shteyner	Partial Orders Based on Inverses Along Elements	Journal of Mathematical Sciences	GI, Green's Rel., Struct. Sem.	3, 6, 18
[79]	2019	A. Guterman and P. Shteyner	On Hartwig–Nambooripad orders	Semigroup Forum	GI, Green's Rel., Struct. Sem.	3, 6, 18
[160]	2019	P. Patrício	Group-regular rings	FILOMAT	Group inverse, Green's Rel., Struct. Rings	11, 22
[161]	2020	P. Patrício	Characterizations of special clean elements and applications	FILOMAT	GI, Ring El.	11, 20, 22, 23
[153]	2020		Characterizations of clean elements by means of outer inverses in rings and applications	J. Algebra Appl.	GI, Green's Rel., Ring El.	3, 5, 7, 8, 11, 12, 20, 21, 22, 23
[139]	2021		n-chained semigroups and $n/2$-perspective modules and rings	to appear in Comm. Algebra	Group inverse, Green's Rel., Struct. Sem., Struct. Rings	2, 16, 24
[141]	2021		Special clean elements, perspective elements and perspective rings	to appear in J. Algebra Appl.	GI, Green's Rel., Ring El.	12, 16, 20, 22, 23, 24
[154]	2021		(b, c)-inverse, inverse along and element, and the Schützenberger category of a semigroup	Categories and General Algebraic Structures with Applications	GI, Green's Rel., Categories	3, 4, 5, 13
[156]	2021	P.P. Nielsen	IC rings and transitivity of perspectivity	J. Algebra Appl.	Struct. Rings	1, 8, 16, 24
[117]	2021	D. Khurana and P.P. Nielsen	Idempotent chains and bounded generation of SL_2	submitted to J. Pure Appl. Algebra	Struct. Rings	16, 24
[140]	2021		Rings with transitive chaining of idempotents	submitted to Proc. of NCRA, VII	Struct. Rings	24

GI stands for *generalized inverses* – Green's Rel. for *Green's relations* – Ring El. for *Ring elements* – Struct. Sem. for *structure of semigroups* – Struct. Rings for *structure of rings.* (Articles are on `http://xmary.perso.math.cnrs.fr`)

List of Symbols

(T, X) (resp. (X, S))	left (resp. right) semigroup act
(T, X, S)	semigroup biact
$<^{\#}$	star partial order
$<^{\dagger}$	dagger/Drazin partial order
$<^{*}$	star partial order
$<^{-}$	minus/Hartwig partial order
$<^{\Theta}$	Mitra and Hartwig partial order
$<_{\mathcal{HL}}$	Hartwig and Luh partial order
$<_{\mathcal{M}}, \leq$	Mitsch partial order (or natural partial order)
$<_{\mathcal{N}}$	Nambooripad partial order
$\bar{e}$	$= 1 - e$, complementary idempotent of e
$\mathcal{M}R$	monoid part of the ring R
$\mathcal{M}$	monoid
$\mathcal{T}(X)$	set of transformation on X
$\mathrm{End}(M)$	endomorphism ring of M
$\mathrm{End}\,(X, S)$	Endomorphism monoid of the right act (X, S)
$\mathrm{End}^{op}\,(T, X)$	Endomorphism monoid of the left act (T, X)
$\hat{\mathfrak{R}}$	unitization of the general ring $\mathfrak{R}$
$\mathrm{im}(a)$	image of the endomorphism a
$\ker(a)$	kernel of the endomorphism a
$\leq$, or ω	natural partial order (on idempotents)

$\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{H}}$	Green's preorders
$\leq_{\widetilde{\mathcal{L}}_E}, \leq_{\widetilde{\mathcal{R}}_E}, \leq_{\widetilde{\mathcal{H}}_E}$	Green's extended preorders (relative to $E \subseteq E(S)$)
$\mathrm{per}(R)$	set of perspective elements of R
$\Re$	general ring
$\Re^*$	adjoint semigroup (monoid) of the general ring $\Re$
$\Re^\circ$	circle semigroup (monoid) of the general ring $\Re$
$\mathrm{reg}(R)$	set of regular elements of R
$\mathrm{reg}(S)$	set of regular elements of S
$\sim_\ell$	left association
$\sim_r$	right association
$\mathrm{sp.\,cl}(R)$	set of special clean elements of R
$\mathbf{RegX}$	regular representation of the biact $\mathbf{X} = (T, X, S)$
$\mathrm{U}(\mathcal{M}), \mathcal{M}^{-1}$	group of units of $\mathcal{M}$
$\mathrm{U}(R), R^{-1}$	group of units of R
$\mathrm{ureg}(\mathcal{M})$	set of unit-regular elements of $\mathcal{M}$
$\mathrm{ureg}(R)$	set of unit-regular elements of R
$\widetilde{\mathcal{L}}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}, \widetilde{\mathcal{J}}$	Green's extended relations (relative to $E = E(S)$)
$\widetilde{\mathcal{L}}_E, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{H}}_E, \widetilde{\mathcal{D}}_E, \widetilde{\mathcal{J}}_E$	Green's extended relations (relative to $E \subseteq E(S)$)
$\mathbb{Z} \oplus \Re$	Dorroh extension of the general ring $\Re$
A''	bicommutant (double commutant, double centralizer) of A
$a^\#$	group inverse of a
$a^\dagger$	Moore-Penrose inverse of a
$A \sim_\oplus A'$	A, A' are perspective
$A \subseteq^\oplus M$	A is a direct summand of M
a^D	Drazin inverse of a
$a^{-(b,c)}$	(b, c)-inverse of a
$a^{-(e,f)}$	Bott-Duffin (e, f)-inverse
a^{-d}, $a^{\Vert d}$	inverse of a along d
a^{-e}	inverse of a along the idempotent $e \in E(S)$, Bott-Duffin inverse of a relative to the idempotent e

a^{-M}	Maximal or natural (generalized) inverse
a^{gD}	generalized Drazin inverse of a
$E(R), \mathrm{idem}(R)$	Set of idempotents if R
$E(S)$	set of idempotents of S
$H(S)$, $S^{\#}$	set of completely regular elements of S
$I(a)$, or $A(a)$	set of inner inverses (associates) of $a \in S$
$J(R)$	Jacobson radical of R
$L(C)$	semigroup of left translations on C
L_a, R_a, J_a, D_a, H_a	Equivalence classes of a for Green's relations
M	Module
$N(R)$	set of nilpotent elements of R
p	spectral idempotent, spectral projection
$Q(\Re)$	set of quasiregular elements of $\mathcal{R}$
R	ring
$R(C)$	semigroup of right translations on C
$r_R(a)$	right annihilator of the ring element $a \in R$
S	semigroup
$S \trianglelefteq T$	S is a subsemigroup of T
S^1	monoid generated by S
$V(a)$	set of reflexive inverses of $a \in S$
$W(a)$	set of outer (weak) inverses of $a \in S$
$Z(A)$, A'	commutant (centralizer) of A
$\mathcal{L}$,$\mathcal{R}$,$\mathcal{J}$,$\mathcal{D}$, $\mathcal{H}$	Green's relations
SemBiact	Category of semigroup biacts

I

The basics... and some inspiration

Part I - The basics... and some inspiration

In this part, I first recall in Chapter 1 the basics of generalized inverses, semigroups and rings that will be used in the memoir. I also introduce the necessary notations. Second, I present thoroughly in Chapter 2 the (long-studied) interplay between the group inverse, completely regular elements and semigroups, and strongly regular rings. In particular the now very-well known structure of such semigroups and rings is exposed. Not only does this example serve as an introduction to my work, but it also explains more generally the kind of mathematics that inspired me, the problems I aim to tackle and the type of results I try to obtain.

Chapter 1

Definitions and notations

In this memoir, S is a semigroup and S^1 denotes the monoid generated by S. By $E(S)$ we denote the set of idempotents, and by $Z(E(S)) = \{x \in S | xe = ex \ (\forall e \in E(S))\}$ its centralizer. More generally, the commutant of a set $A \in \mathcal{P}(S)$ will be denoted by $Z(A)$, or more often A'. The bicommutant (also called double commutant) of A will always be denoted by A''. Monoids will be denoted by $\mathcal{M}$ and their group of units by $\mathrm{U}(\mathcal{M})$ or $\mathcal{M}^{-1}$, whereas M will denote a right k-module, with k a given ring the will depend on the context (in general, k will remain unspecified). Rings are associative, non-commutative and unital unless otherwise stated, and denoted by R. To any module M is attached its endomorphism ring $\mathrm{End}(M)$, and conversely any ring R defines a right module R_R and a left module ${}_RR$ over $k = R$. When R is unital, R and $\mathrm{End}(R_R)$ are isomorphic. Endomorphisms will be written on the left, so that the *image* of $a \in \mathrm{End}(M)$ is $\mathrm{im}(a) = aM$ and its *kernel* is $\ker(a) = \{x \in M | ax = 0\}$. The corresponding notions for elements of a ring are the *right principal ideal generated by a* aR and the *right annihilator of a* $r_R(a) = \{x \in R | ax = 0\}$. A ring with or without an identity will be called a general ring, and denoted by $\Re$. Attention to general rings has grown up lately, and (probably due to my semigroup-oriented mind), I tried in my research to extend notions *à priori* defined only in unital rings to general rings when possible. All semigroup definitions and results apply to any unital (resp. general) ring R by considering its monoid (resp. semigroup) part $\mathcal{M}(R)$ (resp. $S(R)$). In this case, we will however denote more simply its set of idempotents by $E(R)$ or $\mathrm{idem}(R)$, and its group of units by $\mathrm{U}(R)$ or R^{-1}.

We assume the reader familiar with the fundamentals of semigroup theory, as found for instance in [98]*, and module and ring theory, as found in [127]*and [128]*.

However, since they are a crucial part of my research, I recall below some specific notions: *Green's relations* and *generalized inverses* (semigroup theory), and *internal cancellation* and *perspective modules* (module theory).

1.1) Green's relations - Generalized inverses (Semigroup Theory)

Most of my research makes use of the *Green's preorders* and the *Green's relations* in a semigroup [73]*. For elements a and b of S, Green's preorders $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{H}}$ are defined by

$$\begin{aligned} a \leq_{\mathcal{L}} b &\iff S^1a \subseteq S^1b \iff \exists x \in S^1,\ a = xb; \\ a \leq_{\mathcal{R}} b &\iff aS^1 \subseteq bS^1 \iff \exists x \in S^1,\ a = bx; \\ a \leq_{\mathcal{J}} b &\iff S^1aS^1 \subseteq S^1bS^1 \iff \exists x, y \in S^1,\ a = xby; \\ a \leq_{\mathcal{H}} b &\iff (a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b)\,. \end{aligned}$$

If $\leq_{\mathcal{K}}$ is one of these preorders, then $a\,\mathcal{K}\,b \iff \{a \leq_{\mathcal{K}} b \text{ and } b \leq_{\mathcal{K}} a\}$, and $K_a = \{b \in S,\ a\,\mathcal{R}\,b\}$ denotes the $\mathcal{K}$-class of a. Finally, we define the relative product $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. As the relations $\mathcal{L}$ and $\mathcal{R}$ commute, then $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \vee \mathcal{R}$ and it is an equivalence relation. In any semigroup, $\mathcal{D} \subseteq \mathcal{J}$, but equality will hold in the interesting class of *stable semigroups*. Among the main properties of the preorder $\leq_{\mathcal{L}}$ are right congruence ($(\forall x \in S)\ a \leq_{\mathcal{L}} b \Rightarrow ax \leq_{\mathcal{L}} bx$) and right cancellation ($(\forall x, y \in S^1)\ a \leq_{\mathcal{L}} b \wedge bx = by \Rightarrow ax = ay$). Dually for $\leq_{\mathcal{R}}$. It follows that $\mathcal{L}$ is a right congruence, and $\mathcal{R}$ is a let congruence. Relation $\mathcal{H}$ is not a congruence in general.

Of crucial importance in my research is the following result due to Miller and Clifford [166, Theorem 3]*(see also [98, Proposition 2.3.7]*), that relates Green's relations and existence/location of idempotents.

Theorem 1.1.1 ([166, Theorem 3]*). Let $a, b \in S$. Then $ab \in R_a \cap L_b$ iff $L_a \cap R_b$ contains an idempotent. If this be the case then

$$aH_b = H_ab = H_aH_b = H_{ab} = R_a \cap L_b.$$

(This idempotent, if it exists, is unique, for a $\mathcal{H}$-class contains at most one idempotent). Whenever, $ab \in R_a \cap L_b$, we say that ab is a *trace product* (for they are the non-zero products in the *trace of the semigroup*, see [166]*).

Moreover, Green's relations take an interesting form when applied to idempotents (the first occurrence is probably [40]*, and the restrictions of Green's preorders to idempotents are notably a primitive notion regarding *biordered sets* ([60]*, [177]*, [178]*, [188]*). For any two idempotents $e, f \in E(S)$, $e \leq_{\mathcal{L}} f$ iff $ef = e$ and dually $e \leq_{\mathcal{R}} f$ iff $fe = e$. The intersection $\leq_{\mathcal{H}}$ of these two preorders, when restricted to idempotents, is actually a partial order (any two $\mathcal{H}$-related idempotents are equal) called the *natural partial order*, and denoted by $\leq$ afterward. Thus for any two idempotents $e, f \in E(S)$, $e \leq_{\mathcal{H}} f \iff e \leq f \iff e = ef = fe$. It holds that $S^1e = S^1f$ iff $e\,\mathcal{L}\,f$ iff $ef = e, fe = f$, and the relation does not depend on the ambient semigroup. We say that e and f are *left associates* in this case, and we write $e \sim_\ell f$. Dually, we write $e \sim_r f$ to denote that e and f are *right associates*. Finally, e and f are $\mathcal{D}$-related iff

$eS^1 \simeq fS^1$ (as S-sets, and e, f are termed *isomorphic*), iff $e = ab$ and $f = ba$ for some $a, b \in S$ (that we can choose to form a *regular pair*, see below).

Miller and Clifford's theorem on trace products admits the following interpretation when applied to isomorphic idempotents that will prove very useful in many articles. Through probably folklore, I could not find any reference in the literature.

Theorem 1.1.2 ([156, Proposition 2.2])**.** Let S be a semigroup and $e, f \in E(S)$ be $\mathcal{D}$-related idempotents with $e = ab, f = ba$ for some $a, b \in S$. Then eae is invertible in eSe (equiv. faf is invertible in fSf) iff there exists $h \in E(S)$ such that $e \sim_r h \sim_\ell f$.

We now define *regularity* and *generalized inverses*. Regularity was first defined and studied by John Von Neumann in the context of rings, in relation with his axiomatisation/coordinatization of continuous geometries via lattices [179]*; indeed, he proved that every complemented modular lattice (with a homogeneous basis of at least four elements) is isomorphic to the lattice L of finitely-generated submodules of the left R-module R^n, for some regular ring R (and conversely any such lattice is completemented modular). In the context of semigroups, it was soon recognize that the abundance of idempotents in regular semigroups made their study easier, and most of the first results in semigroup theory dealt only with regular semigroups.

We say a is *(von Neumann) regular* in S if $a \in aSa$. The set of regular elements of S will be denoted by $\mathrm{reg}(S)$. A particular solution to $axa = a$ is called an *inner inverse*, or *associate*, of a. A solution to $xax = a$ is called an *outer inverse* (or *weak inverse*). Finally, an element that satisfies $axa = a$ and $xax = x$ is called a *reflexive inverse* of a. The set of all inner (resp. outer, resp. reflexive) inverses of a is denoted by $I(a)$ (or $A(a)$) (resp. $W(a)$, resp. $V(a)$). If $b \in V(a)$, we also say that (a, b) is a *regular pair*. If $b \in I(a)$, then (a, bab) is always a regular pair. The study of specific inner/outer/reflexive inverses via equations is the realm of generalized inverse theory. Note the conceptual distinction between the two notions; whereas regularity deals with **elements in specific locations** ($a \in aSa$, $a \in R_e$ for some idempotent $e \in E(S)$,...), generalized inverses deal with the **solutions** $x \in S$ **to equations** of the form $axa = a$, $xax = x$, ... for some given $a \in S$. Observe also the apparition of idempotents by computing $(ab)^2 = ab = e, (ba)^2 = ba = f$ when $aba = a$ or $bab = b$. Actually, regularity admits a description in terms of Green's relations and idempotents: a is regular iff $a \,\mathcal{R}\, e$ (equiv. $a \,\mathcal{L}\, f$) for some idempotent $e \in E(S)$ (equiv. $f \in E(S)$); the semigroup S itself is regular iff each $\mathcal{D}$-class contains an idempotent.

In the presence of a identity (monoid $\mathcal{M}$, in particular the monoid part of a ring), we will be interested in *unit-regular* elements, those regular elements $a \in \mathcal{M}$ that admit a unit (a.k.a. invertible) inner inverse $u \in V(a) \cap \mathrm{U}(\mathcal{M})$. By $\mathrm{ureg}(\mathcal{M})$ we denote the set of unit-regular elements of $\mathcal{M}$.

A regular element $a \in S$ is *completely regular* if a lies in a subgroup of S, or equivalently (see Chapter 2) if there exists an inner inverse x of a that commutes with a. In this case, $b = xax \in V(a)$ and commutes with a. A commuting reflexive inverse, if it exists, is unique and denoted by $a^\#$. It is usually called the *group inverse* of a, and completely regular elements are also called group invertible elements. In the following, $H(S)$ will denote the set of completely regular (group invertible) elements (also denoted by $S^{\#}$ in some papers).

There are two more very classical generalized inverses. The first one is the *Moore-Penrose inverse* [173]*, [189]*, which is defined in any $*$-semigroup (semigroup with involution). The Moore-Penrose inverse of $a \in S$, if it exists, is (the only) reflexive inverse x of a such that additionally the associated idempotents ax and xa are projections: $(ax) = (ax)^*$ and $(xa)^* = xa$. It is denoted by $a^\dagger$. The second one is the *Drazin inverse* [50]*. The Drazin inverse of $a \in S$, denoted by a^D (if it exists) is the only outer inverse x of a that commutes with a and satisfies $a^{n+1}x = a^n$ for some $n \in \mathbb{N}$. The smallest such n is called the Drazin index. An element $a \in S$ is then group invertible iff it is Drazin invertible with index 0 or 1. In this case $a^{\#} = a^D$. If a is Drazin invertible then one also says that a is *completely π-regular* (for this happens iff some power of a is completely regular).

1.2) Internal cancellation - Perspectivity (Module Theory)

Regarding modules, apart the general theory, two less known notions will be important: *Internal Cancellation*, and *Perspectivity*.

Informally, cancellation of modules asks the following question: if $A \oplus B \simeq A \oplus C$, does $B \simeq C$? This question is related to Dedekind-finiteness, substitution property, finite exchange property and internal cancellation, and proved a fruitful area of research in module and ring theory. We consider here *modules with internal cancellation* (IC modules), see [63]*, [75]*, [110]*. A module M satisfies internal cancellation (IC), if whenever $M = A \oplus B = A' \oplus B'$, then $A \simeq A'$ implies $B \simeq B'$.

Another property is perspectivity. Two direct summands $A, A' \subseteq^{\oplus} M$ of a module M are *perspective* (denoted by $A \sim_{\oplus} A'$) if they have a common complementary summand in M: $A \oplus B = A' \oplus B = M$ for some $B \subseteq^{\oplus} M$. A module M is perspective if any two isomorphic direct summands of M are perspective.

Perspectivity is actually a more general notion and can be defined in any complemented lattice. Its use in module theory traces back to J. Von Neumann in the 40's (he worked on the modular lattice of principal ideals of a regular ring). It has then been reconsidered in the 60's and 70's by L. Fuchs [68]*and D. Handelman [82]*, in link with cancellation and substitution properties. The study of perspective modules in full generality is much more recent, and due to Garg et al. in 2014 [69]*.

Apparently, these notions are far from the considerations of the previous section, that dealt with regularity and generalized inverses. Next result, which appears in [141], explicits the link between complementary summands and reflexive inverses. It is a variation on the following (folklore) result: direct sum decompositions $A \oplus A' = M$ are in bijective correspondence with idempotents of $R = \mathrm{End}(M)$, where the idempotent is the projection on A parallel to A'.

Lemma 1.2.1 ([141]). Let A in M be a direct summand. Then $A = aM$ for some regular element $a \in R = \mathrm{End}(M)$. Moreover, any complementary summand of $A = aM$ is of the form $(1 - ab)M = r_M(b)$ for some $b \in V(a)$ (where $r_M(b) = \{x \in M | bx = 0\}$ is the right annihilator of b).

Thus direct summands and their complementary summands, idempotents, and regular elements and their reflexive inverses may be seen as different models of the same abstract notion. Chapter 24 is an exposure of the work of my co-authors and myself based on this simple consideration.

Chapter 2

Complete regularity - or the mathematical background that inspired me

In this section, I will try to describe what I like in mathematics, and what I intend to do in my research, by describing some mathematical results that inspired me: the notion of complete regularity.

At the start, we have a single notion, that of a commuting reflexive inverse of an element (that is basically only three semigroup equations: $axa = a, xax = x, ax = xa$). **But from these simple equations, mathematicians have been able to build new concepts and new theories**, by interpreting the notion in different ways, and at different levels (element-wise, globally). Also, depending on the environment (semigroups, rings, modules) new tools will be at hand, and the similarities and differences that will appear will shed a new light on the structures at play.

2.1) Element-wise characterizations

Let $a \in S$. Recall that a commuting reflexive inverse of a, if it exists, is unique and denoted by $a^{\#}$, and that an element admitting a commuting reflexive inverse is called completely regular.

The following characterization of group invertibility in terms of Green's relation $\mathcal{H}$ and inverses ([73, Theorem 7]*, [98, Theorem 2.2.5 and 2.3.4]*and [166, Corollaries 3 and 4]*) is a cornerstone of many results in semigroup theory, and was very inspirational for my research. I actually rediscovered the first equivalence of Theorem 2.1.1 in [145, Theorem 2] without any knowledge of semigroups theory nor of Green's relations. Indeed, at this very moment I had been working on the functional analysis background of statistical theory (PhD Thesis and subsequent work [198], [155], [142], [187], [199], [143], [29]), and just slightly moved from function spaces to operator and C^*-algebras. Forgetting all the topological and functional properties, and keeping only the necessary and sufficient - as it happened algebraic- conditions was the starting point of my research in generalized inverses, semigroup theory and non-commutative algebra in general.

Theorem 2.1.1 ([73, Theorem 7]*, [166, Corollaries 3 and 4]*)**.** Let a, a' be elements of a semigroup S. Then
(1) $a^{\#}$ exists iff $a \,\mathcal{H}\, a^2$ iff H_a is a group iff $a \,\mathcal{H}\, e$ for some idempotent $e \in E(S)$.
(2) Assume (a, a') be a regular pair. Then $aa' = a'a$ if and only if $a \,\mathcal{H}\, a'$.
(3) If H is a $\mathcal{H}$-class of S, then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S.

(The last statement is usually known as Green's Theorem).

In particular, the maximal subgroups of S coincide with the $\mathcal{H}$-classes of idempotents, which are pairwise disjoint. An element is then completely regular iff it belongs to some subgroup of the semigroup (the maximal subgroup with this property being $H_a = H_e$, $\mathcal{H}$-class of a, with identity $e = aa^{\#} = a^{\#}a \in E(S)$). This is the reason why completely regular elements are also called group invertible elements (or sometimes simply group elements), and the commuting reflexive inverse the *group inverse*. Also, a is completely regular iff it is invertible in some local submonoid eSe.

Consider now the case of a unital ring. First, we deduce directly that elements with a commuting reflexive inverse coincide with the strongly regular elements of ring theory (where $a \in R$ is *strongly regular* if $a \in a^2R \cap Ra^2$), or with elements invertible in some corner ring $eRe, e \in E(R)$. Second, in this case, we can characterize them by means of units: $a \in R$ is strongly regular iff $u = 1 + a - aa'$ is a unit, for some (all) inner inverse of a. And third, assume that $R = \text{End}(M)$ is the endomorphism ring of some (right k-)module $M = M_k$. Then $\varphi \in \text{End}(M)$ is strongly regular iff $\text{ran}(\varphi) \oplus \ker(\varphi) = M$.

Finally, starting from a simple commutation property for reflexive inverses, we made connections with:

(1) equality of left/right ideals generated by a and a^2, or a and an idempotent;

(2) maximal subgroups;

(3) invertible elements in local submonoids (or corner rings);

(4) units (in ring theory);

(5) direct sum decompositions (in module theory).

Understanding notions in different ways, as in this example, serves as a strong guideline for my research.

2.2) Structure theorems

Even more interesting are the global results. By definition, a semigroup (resp. ring) is completely regular (resp. strongly regular) if all its elements are completely regular (resp. strongly regular).

From Theorem 2.1.1, we deduce that a semigroup S is completely regular iff it is a (disjoint) union of groups (its $\mathcal{H}$-classes). However, a more interesting decomposition holds. If S is completely regular, then Green's relation $\mathcal{J}$ is a semilattice congruence ($S \backslash \mathcal{J}$ is a semilattice), $\mathcal{J} = \mathcal{D}$ and each $\mathcal{J}$-class is a *completely simple semigroup*. Thus S **is a semilattice of completely simple semigroups, whose structure is well-known thanks to Rees Theorem: a semigroup is completely simple iff it is (isomorphic to) a Rees matrix semigroup over a group.**

If we consider the case of rings (which are much more rigid than semigroups), then the situation changes drastically. In this case, each element has a unique reflexive element (we say that the ring is *inverse*), idempotents of the ring are central and the (monoid part of the) ring is a semilattice of groups. Also, this happens iff the ring is regular and a subdirect product of division rings. Moreover, it was proved by Arens and Kaplansky that this is equivalent with the one-sided property $a \in a^2R$ ($\forall a \in R$). Precisely, the following are equivalent:

Theorem 2.2.1 (*)**.** Let M be a module, $R = \mathrm{End}(M)$ its endomorphism ring and $\mathcal{M}R$ the monoid part of R. The the following statements are equivalent:

(1) $(\forall a \in R)\ a \in a^2R$ (this is the original definition of a strongly regular ring);

(2) R is regular and for any $a \in R$ and some (every) regular pair (a, a') then $1 + a - aa'$ is a unit;

(3) R is regular and reduced (the set of nilpotent elements reduces to 0, $N(R) = 0$);

(4) R is regular and a subdirect product of division rings;

(5) Every principal left ideal of $\mathcal{M}R$ is generated by a central idempotent;

(6) $\mathcal{M}R$ is completely regular and inverse;

(7) $\mathcal{M}R$ is regular and idempotents are central;

(8) $\mathcal{M}R$ is a semilattice of groups;

(9) $(\forall \varphi \in \mathrm{End}(M)), \mathrm{ran}(\varphi) \oplus \ker(\varphi) = M$;

(10) $R = \mathrm{End}(M)$ is regular and direct summands of R are uniquely complemented;

(11) $R = \mathrm{End}(M)$ is regular and direct summands of R are *fully invariant* (if N is a direct summand, then $\varphi(N) \subseteq N (\forall \varphi \in \mathrm{End}(M))$).

(The last result is [139, Corollary 4.8]).

Theorems of this kind, that relate properties of regular elements, idempotents, principal ideals, units (in the case of rings), direct summands (in the case of modules) and global structures are one of the ultimate goals of my research.

II

The inverse along an element

Part II - The inverse along an element

The inverse along an element was defined in [146], and, since then, it has been thoroughly studied by many scholars. In this part, we first introduce this notion and study its very general properties (in Chapter 3). Its relation to more common generalized inverses (group, Drazin or Moore-Penrose inverses) is exposed in Chapter 4. In this chapter, we also explore inverses along centralizers and idempotents, with a particular emphasis on commuting and bicommuting idempotents. Cline's formula for such inverses is also discussed at the end of the chapter. Then, the link with the the (b, c)-inverse of Drazin [51]*is studied in Chapter 5, through a categorical interpretation of the inverse along an element. Reverse order laws are also studied in this manner. In Chapter 6, we study partial orders based on the inverse along an element and finally, Chapter 7 presents the additional properties of the inverse along an element specific to the ring case.

Chapter 3

The inverse along an element

When I first learn about generalized inverses, they were different competitive notions such as the group inverse, the Drazin inverse or the Moore-Penrose inverse (and later other notions such as the core or dual core inverse [197]*also appeared). And each of these inverses had its specific features and studies. However, I noticed in my early papers on the subject [145] and [144], that some of their properties shared certain similarities. Also, while these two first papers dealt with operator algebras, I noticed that the functional and topological assumptions played essentially no role (except for certain existence criteria), and most of the properties could be expressed algebraically by using the product operation only. This aroused my curiosity and ultimately, lead to the introduction of the *inverse along an element* in a semigroup [146], that encompass all the previous notions and allow for a unified treatment of some of their main properties. As we will see shortly, this inverse along an element makes great use of Green's relations.

Since 2011, this notion has further been developed and studied by the present author and his coauthors [147], [157], [158], [151], [225], [78], [153], [154] as well as by many others mathematicians, in various settings such as semigroups [223]*, [224]*, rings [11]*, [37]*, [104]*, [107]*, [197]*, [226]*, [222]*, [227]*, matrices over fields [36]*, [211]*, tensors [203]*, [175]*or C^*, Banach and Operator algebras [12]*, [19]*, [18]*, [39]*, [23]*, [105]*(this list being by no means exhaustive).

It must be noted that concomitantly and independently to my research [146], M.P. Drazin introduced the (b, c)-inverse [51]*, which has also been widely studied afterward [13]*, [35]*, [52]*, [53]*, [54]*, [55]*, [56]*,, [108]*, [208]*, [215]*. It happens however that the (b, c)-inverse is equivalent to the inverse along an element (see notably [218, Lemma 2.13]*and the papers [153], [154], or directly Chapter 5 in the present memoir). However, this equivalence seems unknown by many scholars, thus leading to the duplication of many results. This is one of the reason of the redaction of [154].

3.1) Definition and characterizations

Lemma 3.1.1 ([146, Lemma 3 and Theorem 6])**.** Let $a, b, d \in S$. Then the following statements are equivalent:
1. $bad = d = dab$ and $b \leq_{\mathcal{H}} d$.
2. $bab = b$ and $b\mathcal{H}d$.

Moreover, if such a b exists, it is unique.

Definition 3.1.2 ([146, Definition 4])**.** Let S be a semigroup and $a, d \in S$. We say that $b \in S$ is **the inverse of** a **along** d, denoted by a^{-d} (or $a^{\|d}$) if it satisfies one of the equivalent statements of Lemma 3.1.1. If it satisfies additionally that $aba = a$ then it is the *inner inverse of* a *along* d.

The notation $a^{\|d}$ was suggested by R. Hartwig and used in my first papers on the subject [146], [157], [158], [225]. Then, it was suggested by a referee to use the notation a^{-d} (see explanation below in section 4.1), a notation that I found actually more convenient and tried to use ever since. However, some other authors still use the former notation $a^{\|d}$. In some papers, the inverse along an element is also called *Mary inverse* (along an element).

It follows from the definition that the inverse along an element may be seen as a parametrized outer inverse. This is used notably in relation with partial orders in [78] and [77]*. Observe also that the parameter d must be regular in order that the inverse along d exists.

It is of crucial importance to observe that, while the first characterization of Lemma 3.1.1 : $bad = d = dab$ and $b \leq_{\mathcal{H}} d$ makes a full use of the element d, the second characterization: $bab = b$ and $b\mathcal{H}d$ states exactly that a^{-d} **is the unique outer inverse of** a **in the** $\mathcal{H}$**-class** H_d **of** d. Thus, the inverse along an element is actually more an *inverse along an* $\mathcal{H}$*-class* (for an implication of this result, see Section 4.2). This is for instance stated explicitly in [158]. Since an $\mathcal{H}$-class is by definition the intersection of a $\mathcal{R}$-class and a $\mathcal{L}$-class, it follows that the inverse along d may be defined using the $\mathcal{R}$-class and the $\mathcal{L}$-class of two elements $b, c \in S$ such that $d \in R_b \cap L_c$. What we obtain is precisely the (b, c)-*inverse* of Drazin [51]*(see Section 5.1).

3.2) General properties

Despite its very general definition, the inverse along an element still has many very interesting properties.

First, there are existence results, and characterizations.

Theorem 3.2.1 ([146, Lemma 3 and Theorems 6,7], [158, Theorem 2.2 and Corollary 2.5], [11, Theorem 8.4]*)**.** Let S be a semigroup and $a, d \in S$. Then the following statements are equivalent:
(1) $bab = b$ for some $b \in H_d$ (a is invertible along d);
(2) $bad = d = dab$ for some $b \leq_{\mathcal{H}} d$;
(3) $ad\,\mathcal{L}\,d$ and H_{ad} is a group;
(4) $da\,\mathcal{R}\,d$ and H_{da} is a group;
(5) $dad\,\mathcal{H}\,d$;
(6) $H_d a H_d = H_d$.
In this case,

$$a^{-d} = b = d(ad)^{\#} = (da)^{\#} d = d(dad)^{-} d$$

for any $dad^{-} \in I(dad)$.

We make some observations:

- if a is invertible along d, then d is $\mathcal{L}$-related to an idempotent (the identity of H_{ad}). Thus d is regular, so that $I(dad)$ is not empty (equivalently, $d = dab = d\left(a(da)^{\#}\right) d$). The equality $a^{-d} = d(dad)^{-}d$ was proved by Benitez and Boasso [11, Theorem 8.4]*, in the context of rings. But their result carries out straightforwardly to semigroups;
- characterizations (3) and (4) show that group inverses are ubiquitous with regard to generalized inverses;
- the equation $dad\,\mathcal{H}\,d$ characterizes a as a kind of "inner inverse of d modulo $\mathcal{H}$", a statement we took literally and studied carefully in [148] (see Chapter 14). Equivalently, d may be interpreted as an "outer inverse of a modulo $\mathcal{H}$", a direction followed by Fan et al. [217]*;
- The equality $H_d a H_d = H_d$ claims that $G = H_d$ is a maximal subgroup of the *variant semigroup* $S_a = (S, ._a)$ with multiplication $x ._a y = xay$. Conversely, we can prove that any maximal subgroup G of S_a is of the form H_d, for some d such that a is invertible along d (and the identity of G is a^{-d}).

By [103, Theorem 3]*, the inverse along an element can also be characterized as an outer inverse with prescribed idempotents (in the following sense): $b = a^{-d}$ iff $b \in W(a)$ and $ab = td, ba = dt$ for some $t \in I(d)$.

Second, we can characterize when $a^{-d} \in V(a)$ (that is, when a^{-d} is an inner of a, since it is always an outer inverse of a). This is a direct consequence of Miller and Clifford's theorem 1.1.1.

Theorem 3.2.2 ([146, Corollary 9])**.** Let S be a semigroup and $a, d \in S$. Then the following statements are equivalent:
(1) a is invertible along d and $a^{-d} \in V(a)$ (equiv. $a^{-d} \in I(a)$);
(2) a is invertible along d and d is invertible along a;
(3) ad and da are trace products ($ad \in R_a \cap L_d$ and $da \in R_d \cap L_a$).

And third, there are commutation properties.

Theorem 3.2.3 ([146, Theorem 10]). Let S be a semigroup and $a, d \in S$. Then
(1) $a^{-d} \in \{a, d\}''$ (bicommutant of $\{a, d\}$);
(2) $aa^{-d} \in \{ad\}''$ and $a^{-d}a \in \{da\}''$.

The original proof uses directly the definition of the inverse along an element, but an alternative proof is possible by using the bicommuting property of the group inverse. Indeed, let $a, c, d \in S$ with a invertible along d and $c \in \{a, d\}'$ (the commutant of $\{a, d\}$). Then c commutes with ad hence with $(ad)^\#$, and $a^{-d}c = d(ad)^\# c = dc(ad)^\# = cd(ad)^\# = ca^{-d}$. The second statement does not appear directly in [146], but can be obtained by similar arguments. However, it has been obtained by Drazin [54]*in the context of the (b, c)-inverse.

3.3) Extensions and complements

The *one-sided inverse along an element* is defined in [223]*: an element $a \in S$ is *left invertible* along $d \in S$ if $bad = d, b \leq_{\mathcal{L}} d$ for some $b \in S$ (called a left inverse along d and denote by a_l^{-d}) and dually for right invertibility along d. Left and right inverses along d need not be unique. With H. Zhu, J. Chen and P. Patricio, we proved the following result, that generalizes the case of left and right genuine inverses.

Proposition 3.3.1 ([225, Proposition 2.3]). Let S be a semigroup and let $a, d \in S$ be such that $a \in S$ is left and right invertible along $d \in S$. Then a is invertible along d. Moreover any left and right inverses of a along d a_l^{-d} and a_r^{-d} satisfy

$$a_l^{-d} = a^{-d} = a_r^{-d}.$$

And finally, many scholars gave other characterizations and properties, or studied generalizations/specialization of the inverse along an element. Among all the results obtained, we may single out the following ones (in semigroups).

- Regarding the one-sided inverse along an element, it is notably proved that in a $*$-semigroup, a is left invertible along a^* iff it is right invertible along a^* iff a is Moore-Penrose invertible [223]*. The one-sided inverse along an element is more thoroughly studied in [37]*. The right core inverse studied in [209]*is an instance of such one-sided inverse along an element;

- The complete inverse along an element [210]*, is defined as the unique solution (if it exists) to the system $axd = d = dxa, x \leq_{\mathcal{H}} d$. In [210]*, they proved that this is equivalent with a being invertible along d together with $a^{-d}a = aa^{-d}$, and that this notably implies that $d \in S^\#$ (commutation properties of the inverse along an element had already been studied by Benitez and Boasso [11]*, but in the context of rings).

The inverse along an element has also been used as a tool to investigate some concepts usually based on some specific generalized inverses.

- In [77]*, Marki, Guterman and Shteyner introduce a general notion of quotient ring based on inverses along an element (but their construction is also valid at the level of semigroups). As the classical generalized inverses are special cases of the inverse along an element, the new quotient rings encompass the classical quotient rings constructed using various generalized inverses. Secondly, these new quotient rings can also be viewed as Fountain-Gould quotient rings with respect to appropriate subsets (as inverses along an element can be expressed in terms of group inverses by Theorem 3.2.1).

- In [22]*, Burgos et al. study linear preservers of inverses along an element (linear maps ϕ such that a invertible along d implies $\varphi(a)$ invertible along d). Linear preservers problems are traditionally studied for the genuine inverse, the group inverse or the Drazin inverse (or the whole set of inner or outer inverses).

Chapter 4

Inverses along specific elements in semigroups

4.1) Recovering classical inverses

Let S be a semigroup. We already recalled the definition of the group inverse of $a \in S$, as the unique solution (if it exists) to the three equations $axa = a, xax = x$ and $ax = xa$. In the first part, we also defined the Drazin inverse and the Moore-Penrose inverse. Let us recall these two fundamental notions. To study non-regular elements of S, Drazin [50]*introduced another commuting generalized inverse, which is not inner in general. An element $a \in S$ is *Drazin invertible* if the set of equations

1. $ax = xa$;
2. $a^m = a^{m+1}x$;
3. $x = x^2a$.

admit a solution $b \in S$ for some $m \in \mathbb{N}$. The solution is unique if it exists usually denoted by a^D.

Finally, when S is a endowed with an involution $*$ that makes it an involutive semigroup (or $*$-semigroup), i.e. the involution satisfies $(a^*)^* = a$ and $(ab)^* = b^*a^*$, Moore [173]*and Penrose [189]*studied reflexive inverses x of a with the additional property that $(ax)^* = ax$ and $(xa)^* = xa$. Once again this inverse, if it exists, is unique. It is usually called the *Moore-Penrose inverse* (or pseudo-inverse) of a and denoted by $a^\dagger$. As proved in [146], these classical generalized inverses are actually inverses along a specific element.

Theorem 4.1.1 ([146, Theorem 11])**.** Let S be a semigroup (resp. a monoid in (1), resp. a $*$-semigroup in (4)) and $a \in S$.

(1) a is invertible if and only if it is invertible along 1. In this case the inverse along 1 is inner and coincides with the (genuine) inverse.

(2) a is group invertible if and only if it is invertible along a. In this case the inverse along a is inner and coincides with the group inverse.

(3) a is Drazin invertible if and only if it is invertible along some a^m, $m \in \mathbb{N}$, and in this case the two inverses coincide.

(4) a is Moore-Penrose invertible if and only if it is invertible along a^*. In this case the inverse along a^* is inner and coincides with the Moore-Penrose inverse.

In other words:

$$\begin{aligned} a^{-1} &= a^{\|1}, \\ a^{\sharp} &= a^{-a}, \\ a^{D} &= a^{-a^m} \text{ for some integer } m, \\ a^{\dagger} &= a^{-a^*}. \end{aligned}$$

The first equality explains the choice of the notation a^{-d} as a replacement of $a^{\|d}$.

It is also proved that the *core inverse* and *dual core inverse*, defined in any $*$-semigroup as the solutions to the systems of equations $axa = a, xS^1 = aS^1, S^1x = S^1a^*$ and $axa = a, xS^1 = a^*S^1, S^1x = S^1a$ respectively, are inverses along an element (along aa^* and a^*a respectively [197, Theorem 4.3]*).

One of the main interest of the concept is that, thanks to Theorem 4.1.1, any property of the inverse along an element (as given in Chapter 3.2) then leads to a specific property for the previous inverses for free. For instance, the core inverse of a exists iff $aa^*a^2a^*\mathcal{H}aa^*$ by Theorem 3.2.1 (this statement can be refined under additional assumptions such as $*$-*cancellation*, *the Gelfand-Naimark property* or more generally in C^*-algebras). And if it exists, it commutes with any element that commutes with both a and aa^* thanks to Theorem 13.1.3.

4.2) Inverses along centralizers

A left centralizer (also called a left translation) on S is a map $\sigma : S \to S$ that satisfies $\sigma(ab) = \sigma(a)b$. Right centralizers are defined dually, and a centralizer is both a left and right centralizer.

In [225], H. Zhu, J. Chen, P. Patrício and myself study the relation between a^{-d}, $a^{-\sigma(d)}$ and $(\sigma(a))^{-d}$, and observe that when σ is a bijective centralizer, then $\sigma(d)\mathcal{H}d$. Thus [225, Proposition 3.5], a is invertible along d iff it is invertible along $\sigma(d)$, and $a^{-d} = a^{-\sigma(d)}$ (for the inverse along an element depends only on its $\mathcal{H}$-class). Still [225, Proposition 3.5], this is also equivalent with $\sigma(a)$ being invertible along d, in which case $(\sigma(a))^{-d} = \sigma^{-1}\left(a^{-d}\right)$.

Some additional results in the ring case are presented in Section 7.1. Similar results were obtained by Xu et al. for (b,c)-inverses in [219]*.

4.3) Inverses along ((bi)commuting) idempotents

In this section, we will see that idempotents appear naturally when it comes to commutation properties, a statement that will be made precise below. But inverses along non-commuting idempotents proved also very interesting.

4.3.1) Notations and first results

The first occurrence of an inverse along an idempotent appears actually implicitly in [146], where it said: "We remark that if $da = ad$, the two results i) $a^{-d} \in \{a,d\}''$ and ii) $ad\,\mathcal{L}\,d, da\,\mathcal{R}\,d$ and H_{ad}, H_{da} are groups, then give that $b = a^{-d}$ commutes with a and d and that $H_d = H_{ad}$ is a group." Since H_d is a group, it contains an idempotent $e \in S$ (the identity of the group) and since the inverse along d depends only on its $\mathcal{H}$-class, we obtain that under $ad = da$, then $a^{-d} = a^{-e}$ for some $e \in E(S)$, and a^{-e} commutes with a.

We will see (Theorem 4.3.3) that commuting and bicommuting outer inverses are all special cases of inverses along idempotents (equiv. outer inverses in group $\mathcal{H}$-classes). For the moment, we note that the group inverse and the Drazin inverse are inverses along the idempotents $e = aa^{\#}$ and $f = aa^D$ respectively.

The following lemma regarding inverses along an idempotent is straightforward yet crucial.

Lemma 4.3.1 ([147, Lemma 4])**.** Let S be a semigroup, $a \in S$ and $e \in E(S)$. Then a is invertible along e iff eae is a unit in the local submonoid eSe, in which case

$$a^{-e} = e(ae)^{\#} = (ea)^{\#}e = (eae)^{\#} = (eae)^{-1}_{[eSe]}.$$

As such, the *Bott-Duffin inverse* of a relative to the idempotent f of Khurana et.al. [113, Definition 2.12] is just the same as the inverse of a along f.

As an application consider the reverse order law for the inverse along an element, as studied in [224]*, and let $a,b,d \in S$ such that a,b and ab are invertible along d with $(ab)^{-d} = b^{-d}a^{-d}$. Let also $H = \mathcal{H}_d$. Then $H^2 \cap H \neq \emptyset$ hence by Green's theorem H is a group. Working in the group $H = \mathcal{H}_e$, $e \in E(S)$, group of units of the local submonoid eSe and passing to the inverse in the above equation we obtain that $(ab)^{-e} = b^{-e}a^{-e}$ iff $eabe = eae.ebe = eaebe$. We just proved the following.

Lemma 4.3.2 (unpublished)**.** Let S be a semigroup and $a, b, d \in S$ such that a, b are invertible along d. Then the following statements are equivalent:
(1) ab is invertible along d and $(ab)^{-d} = b^{-d}a^{-d}$;
(2) $d\mathcal{H}e$ for some $e \in E(S)$ and $eabe = eaebe$.

In the general case, if a and b are invertible along $e \in E(S)$ then $b^{-e}a^{-e} = (aeb)^{-e}$ ([224, Corollary 2.21]*or [154, Theorem 3.9 (v)]). More general reverse order laws have been studied in [154], see also Section 5.3.

To study precisely inverses along idempotents, I found convenient to introduce the following sets, for any $a \in S$. The first three were introduced in [147], whereas the latter four were introduced in [151] - with two very distinct objectives: [147] was aimed to extend the *Koliha-Drazin inverse* (a.k.a. *generalized Drazin inverse* [49]*, [119]*, [136]*) to semigroups and consider this extension in rings, while the purpose of [151] was to study Cline's formula and Jacobson's lemma for inverses along (bi)commuting idempotents. Cline's formula for the generalized Drazin inverse has been studied in [174]*.

$$\begin{aligned}
\Sigma_0(a) &= \{e \in E(S) | eae\mathcal{H}e\},\\
\Sigma_1(a) &= \{a\}' \cap \Sigma_0(a),\\
\Sigma_2(a) &= \{a\}'' \cap \Sigma_0(a),\\
\Sigma_R(a) &= \{e \in E(S) | e \in aeS \cap Sae, eae = ae\},\\
\Sigma_L(a) &= \{e \in E(S) | e \in eaS \cap Sea, eae = ea\},\\
\Sigma^{\#}(a) &= \{e \in E(S) | e \in eaeS \cap Seae\},\\
\Sigma(a) &= \{e \in E(S) | e \in aS \cap Sa\}.
\end{aligned}$$

Obviously $\Sigma_0(a) = \Sigma^{\#}(a)$ and by Theorem 3.2.1, $e \in \Sigma^{\#}(a)$ iff a is invertible along e. By [147, Lemma 3] and [151, Lemma 3.4]

$$\Sigma_1(a) = \{a\}' \cap \Sigma(a) = \{a\}' \cap \Sigma^{\#}(a) = \Sigma_R(a) \cap \Sigma_L(a).$$

As any set of idempotents, all these sets are partially ordered by the natural partial order: $e \leq f \iff e = ef = fe$. And more specifically, $(\Sigma_2(a), \leq)$ is a semilattice (commutative band) with $e \wedge f = ef$ (product in S) by [147, Proposition 2] (and we will also denote it by $(\Sigma_2(a), \wedge)$ or $(\Sigma_2(a), .)$ to emphasize either on the min operation or on the product operation rather than on the partial order).

Recall that $W(a)$ is the set of outer (or weak) inverses of a. Following [151], we say that $x \in S$ is a right (resp. left) outer inverse of a if it satisfies $ax^2 = x$ (resp. $x^2a = x$), and we denote the set of right (resp. left) outer inverses of a by $R(a)$ (resp. $L(a)$). We also define $R^{\#}(a) = S^{\#} \cap R(a)$, $L^{\#}(a) = S^{\#} \cap L(a)$, $W_0(a) = W^{\#}(a) = S^{\#} \cap W(a)$, $W_1(a) = \{a\}' \cap W(a)$ and $W_2(a) = \{a\}'' \cap W(a)$.

Next theorem proves that there is a bijective correspondence between completely regular (resp. commuting, resp. bicommuting) outer inverses and (resp. commuting, resp. bicommuting) idempotents below a for the $\leq_{\mathcal{H}}$ preorder, and that it extends to an isomorphisms of posets (resp. semilattices) if one consider $W(a)$ as the set of idempotents of the variant semigroup $(S, ._a)$ with product $x._a y = xay$ (so that $(\forall x, y \in W(a))\ x \leq_a y \iff x = xay = yax$).

Define function

$$\begin{array}{rccc} \tau : & S^{\#} & \longrightarrow & E(S) \\ & x & \longmapsto & xx^{\#} \end{array}$$

Theorem 4.3.3 ([147, Theorem 3], [151, Lemma 3.1], [151, Corollary 3.1])**.** Function τ restricts to:
(1) a bijection τ_a^R from $R^{\#}(a)$ onto $\Sigma_R(a)$;
(2) a bijection τ_a^L from $L^{\#}(a)$ onto $\Sigma_L(a)$;
(3) an isomorphism τ_a^0 of posets from $(W_0(a), \leq_a)$ onto $(\Sigma_0(a), \leq)$;
(4) an isomorphism τ_a^1 of posets from $(W_1(a), \leq_a)$ onto $(\Sigma_1(a), \leq)$;
(5) an isomorphism τ_a^2 of of semilattices from $(W_2(a), ._a)$ onto $(\Sigma_2(a), .)$.
Their reciprocal associate e to a^{-e}.
Also, $\tau_a^R(x) = xx^{\#} = ax$, $\tau_a^L(x) = xx^{\#} = xa$ and $\tau_a^1(x) = xx^{\#} = ax = xa$.

(Actually, the results of [147] and [151] do not cover the case $j = 0$ totally, but it can be proved by the same arguments as the proofs therein).

In summary, function τ_a^j is an isomorphism of posets from $(W_j(a), \leq_a)$ onto $(\Sigma_j(a), \leq)$ for $j = 0, 1, 2$.

4.3.2) Application 1: the natural generalized inverse

Let $a \in S$ be completely regular. Then not only $aa^{\#} \in \Sigma_2(a)$, but $e = aa^{\#}$ is actually **the greatest element of** $\Sigma_2(a)$ with respect to the natural partial order (and this remains true for the completely π-regular elements, see Theorem 4.3.6 below).

Consequently, I proposed in [147] the following definitions.

Definition 4.3.4 ([147, Definition 2])**.** Let S be a semigroup, $a \in S$.

1. Let $j = 0, 1, 2$. The element a is $j-$maximally invertible if the set $\Sigma_j(a)$ admits maximal elements for the natural partial order. Elements a^{-e} where e is maximal are then called $j-$maximal generalized inverses of a.
2. If there exists a greatest element $M \in \Sigma_j(a)$, then we say that a is $j-$naturally invertible, and $b = a^{-M}$ is called the $j-$natural generalized inverse of a.
3. Finally, if a is $2-$naturally invertible, the element $aM = aba$ is called the core of a.

The $2-$natural generalized inverse we will also be simply referred to as *the natural inverse*. The two main properties of the natural inverse are the following:

- if $\Sigma_2(a)$ is distributive, maximal implies natural;
- the natural inverse generalizes the group and Drazin inverse.

Recall that a semilattice is distributive if $e \wedge f \leq x$ implies the existence of e', f' such that $e \leq e',\ f \leq f'$ and $x = e' \wedge f'$.

Proposition 4.3.5 ([147, Proposition 2]). Let $a \in S$. If the semilattice $\Sigma_2(a)$ is distributive, then any $2-$maximally invertible element is naturally invertible.

Theorem 4.3.6 ([147, Theorem 3]). Assume $a \in S$ is Drazin invertible with inverse a^D. Then a is 1 and $2-$naturally invertible with inverse $a^{-M} = a^D$.

The natural generalized inverse has been further studied by Kantún-Montiel in [103]*.

4.3.3) Application 2: Cline's formula for commuting outer inverses

We say that two elements u, v of a semigroup S are *primarily conjugate* if $u = ab, v = ba$ for some $a, b \in S$ [125]*. If $u = e$ and $v = f$ are idempotents, one also says that the idempotents are *isomorphic* (or *Kaplansly equivalent*) since this happens iff $eS^1 \simeq fS^1$ (as right S-acts). In this case one can moreover choose (a, b) a regular pair.

Cline's formula relates to the very general family of properties P (or subsets $\mathcal{P}$ of elements that satisfy P) *invariant by primarily conjugation*: if u satisfies P ($u \in \mathcal{P}$) and v is primarily conjugate to u, then v also satisfy P ($v \in \mathcal{P}$). Indeed, it was observed by Cline in his study of generalized inverses of matrices [42]*, [10]*that ab is Drazin invertible iff ba is Drazin invertible, the relation between the two inverses being $(ba)^D = b[(ab)^D]^2 a$ (and dually). The use of the Drazin inverse is crucial since for the genuine (resp. group) inverse, ab invertible (resp. group invertible) does not imply ba invertible (resp. group invertible).

In the following, we fix $a, b \in S$ and define the function on S $\phi_{b,a} : x \mapsto bx^2 a$, and dually $\phi_{a,b}$. It is straightforward to observe that $\phi_{b,a}$ maps $\{ab\}'$ on $\{ba\}'$ and that

$$\phi_{b,a} : (\{ab\}'', \cdot_{ab}) \to (S, \cdot_{ba})$$

is a morphism.

We first study this map regarding the sets of right weak inverses:

$$R(ab) = \{x \in S | abx^2 = x\} \text{ and } R(ba) = \{x \in S | bax^2 = x\}.$$

or equivalently inverses along an idempotent in $\Sigma_R(ab)$ and $\Sigma_R(ba)$.

Lemma 4.3.7 ([151, Lemma 2.1]). Function $\phi_{b,a}$ maps $R(ab)$ on $R(ba)$ and $R(ab) \cap S^\#$ on $R(ba) \cap S^\#$.

We deduce the following Cline's formula for inverses along an idempotent in Σ_R.

Corollary 4.3.8 ([151, Corollaries 3.2 and 3.3]). If $e \in \Sigma_R(ab)$ then $f = b(ab)^{-e} a \in \Sigma_R(ba)$ with

$$(ba)^{-f} = b\left((ab)^{-e}\right)^2 a$$
$$a(ba)^{-f} = (ab)^{-e} a$$
$$af = ea$$

The idempotents e and f of Corollary 4.3.8 are isomorphic as with the previous notations, $e = a\left(b(ab)^{-e}\right)$ and $f = \left(b(ab)^{-e}\right)a$.

We can now interpret these results in terms of commuting (resp. bicommuting) outer inverses, or equivalently inverses along commuting (resp. bicommuting) idempotents by Theorem 4.3.3. In this case, the map restricts to an isomorphism of posets.

Theorem 4.3.9 ([151, Theorem 2.1 and Corollary 2.2])**.** Function $\phi_{b,a}$ restricts to an isomorphism of posets $j = 1$ (resp. semilattices $j = 2$) from $(W_j(ab), \leq_{ab})$ onto $(W_j(ba), \leq_{ba})$, with reciprocal $\phi_{a,b}$.

But for any $s \in S$, $W_j(s)$ and $\Sigma_j(s)$ are always isomorphic posets (by Theorem 4.3.3). We thus deduce the following Corollary.

Corollary 4.3.10 ([151, Corollary 3.4])**.** Let $u, v \in S$ be primarily conjugate elements. Then the following posets $j = 1$ (resp. semilattices $j = 2$) are isomorphic (with their respective structure):

$$W_j(u) \simeq \Sigma_j(u) \cap \simeq \Sigma_j(v) \simeq W_j(v).$$

Figure 4.1 illustrates Corollary 4.3.10 with commutative diagrams for $u = ab$ and $v = ba$. Each map is an isomorphism of the respective structures ($j = 1, 2$).

$$\begin{array}{ccc} W_j(ab) & \longrightarrow & \Sigma_j(ab) \\ \downarrow & & \downarrow \\ W_j(ba) & \longrightarrow & \Sigma_j(ba) \end{array} \qquad \begin{array}{ccc} x = (ab)^{-e} & \longrightarrow & e = xab = ayb \\ \downarrow & & \downarrow \\ y = bx^2a & \longrightarrow & f = bxa = yba \end{array}$$

Figure 4.1: Isomorphims of Corollary 4.3.10

It may not be clear at first sight, but Corollary 4.3.10 and its associated graphical interpretation Figure 4.1, that explains in details the isomorphisms at stake, is indeed a proper generalization of the original Cline's formula (for Drazin inverses) to the general case of inverses along commuting and bicommuting idempotents.

First, we can rewrite it as in Corollary 4.3.8 (for the right outer inverses).

Corollary 4.3.11. Let $e \in \Sigma_j(ab)$, $j = 1, 2$. Then $f = b(ab)^{-e}a \in \Sigma_j(ba)$ with

$$\begin{aligned} (ba)^{-f} &= b\left((ab)^{-e}\right)^2 a \\ a(ba)^{-f} &= (ab)^{-e}a \\ af &= ea \end{aligned}$$

Second, assume that ab is Drazin invertible with index $n \in \mathbb{N}$. Then by Theorem 4.3.6, $(ab)^D$ is the natural inverse of ab, that is $(ab)^D = (ab)^{-M}$ with M greatest element of $\Sigma_2(ab)$. Thus $f = b(ab)^{-M}a = b(ab)^D a$ is the greatest element of $\Sigma_2(ba)$, and (ba) is naturally invertible with natural inverse $y = b\left((ab)^D\right)^2 a$. By definition it (bi)commutes with ba, is an outer inverse and finally

$$y(ba)^{n+2} = b\left((ab)^D\right)^2 a(ba)^{n+2} = b\left((ab)^D\right)(ab)^{n+1}a = b(ab)^n a = (ba)^{n+1}.$$

Thus ba is Drazin invertible with Drazin inverse $b\left((ab)^D\right)^2 a$.

Observe that, more generally, $x \mapsto bx^2a$ sends the $j-$natural generalized inverse of ab (if it exists) to the $j-$natural generalized inverse of ba ($j = 1, 2$) since the previous isomorphisms preserve the partial order.

Chapter 5

A categorical interpretation of the inverse along an element and the (b, c)-inverse, and Reverse Order Laws

5.1) The (b, c)-inverse vs. the inverse along an element

About the same time of the appearance of the inverse along an element in [146], M. P. Drazin defined [51]*the (b, c)-*inverse*, that can be seen as an extension of the *Bott-Duffin* (e, f)-*inverse* (which is recovered by letting $b = e$ and $c = f$ be idempotents), and that generalizes the classical generalized inverses (group inverse, Moore-Penrose inverse, Drazin inverse).

Definition 5.1.1 ([51]*). Let S be a semigroup and $a, b, c, x \in S$. Then x is a (b, c)-*inverse* of a if
(1) $x \in (bSx) \cap (xSc)$;
(2) $xab = b$, $cax = c$.

A (b, c)-inverse, if it exists, is unique and an outer inverse of a ([51, Theorem 2.1]*). We will denote the (b, c)-inverse by $a^{-(b,c)}$ in the sequel.

In [153, Proposition 1.4] and [154, Theorem 2.4], it is proved that these notions are actually equivalent.

Theorem 5.1.2 ([154, Theorem 2.4])**.** Let S be a semigroup.
(1) Let $a, b, c, x \in S$. If a is (b, c)-invertible with inverse x, then $b\mathcal{D}c$ and for all $d \in R_b \cap L_c$, a is invertible along d with inverse x.
(2) Let $a, d \in S$. If a is invertible along d, then for all $b \in R_d$ and $c \in L_d$, a is (b, c)-invertible and $a^{-(b,c)} = a^{-d}$.
(3) In particular, if $a, d \in S$ are such that a is invertible along d, then $e = a^{-d}a$ and $f = aa^{-d}$ are idempotents such that $e \leq_{\mathcal{R}} d$ and $f \leq_{\mathcal{L}} d$. But also $ed = d$ and $df = f$ by definition of the inverse along d, and $e \in R_d$, $f \in L_d$. Finally a is Bott-Duffin (e, f)-invertible and $a^{-(e,f)} = a^{-d}$ by (2).

Consequently, we see that the requirements in the definition of the (b, c)-inverse can be relaxed, and that this gives a very simple existence criterion.

Corollary 5.1.3 ([154, Corollary 2.5. and Theorem 2.6. (or 2.7.)])**.** Let S be a semigroup and $a, b, c, x \in S$. The following statements are equivalent:
(1) x is the (b, c)-inverse of a;
(2) $xab = b, cax = c$, $x \leq_{\mathcal{R}} b$ and $x \leq_{\mathcal{L}} c$;
(3) $xax = x$ and $x \in R_b \cap L_c$.
This happens iff $cab \in R_c \cap L_b$.

As a consequence of Corollary 5.1.3, let bc be a trace product ($bc \in R_b \cap L_c$). Then a is (b, c)-invertible iff a is invertible along bc, in which case $a^{-(b,c)} = a^{-bc}$. As another application, assume that a is invertible along d and recall the following equality $a^{-d} = d(dad)^{-}d$ due to [11, Theorem 8.4]*. Let b, c such that $d \in R_b \cap L_c$, $d = bx = yc$ for some $x, y \in S^1$. Then by cancellation properties, as $dad(dad)^{-}(dad) = dad$ then $cad(dad)^{-}dab = cab = cab\,(x(dad)^{-}y)\,cab$ and $(cab)^{-} = (x(dad)^{-}y)$ is an inner inverse of cab. Finally

$$a^{-(b,c)} = a^{-d} = d(dad)^{-}d = b\left(x(dad)^{-}y\right)c = b(cab)^{-}c.$$

While I mainly use the inverse along an element in my research, at some places it has been useful to consider the (b, c)-inverse (or more precisely the (e, f)-inverse, $e, f \in E(S)$), such as in [153] and [154].

5.2) Miller and Clifford's theorem revisited

Recall that Miller and Clifford's theorem [166, Theorem 3] (Theorem 1.1.1) states that ab is a trace product ($ab \in R_a \cap L_b$) iff the $\mathcal{H}$-class $H = L_a \cap R_b$ contains an idempotent. We extend this result, and provide applications to the inverse along an element, the (b, c) inverse and the Bott-Duffin inverse.

Theorem 5.2.1 (unpublished)**.** Let S be a semigroup, $a, b \in S$ and $z \in S^1$. Let also $c \in L_a \cap R_b$. Then the following statements are equivalent:
(1) $azb \in R_a \cap L_b$;
(2) $czc\mathcal{H}c$;
(3) $azc\mathcal{H}a$;
(3′) $czb\mathcal{H}b$;
(4) $az\mathcal{R}a$ and $L_{az} \cap R_b$ contains an idempotent;
(4′) $zb\mathcal{L}b$ and $L_a \cap R_{zb}$ contains an idempotent.

Proof. Exchanging the roles of a and b, (1) and (2) are self-dual whereas (3) and (3′) (resp. (4) and (4′)) are dual statements. We prove that (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) and that (2) $\Rightarrow$ (4) $\Rightarrow$ (1). As $c \in L_a \cap R_b$ then $c = xa = by$ and $a = uc, b = cv$ for some $x, y, u, v \in S^1$.

(1) $\Rightarrow$ (2) Assume that $azb \in R_a \cap L_b$. As $azb\mathcal{R}a$ then by left congruence $xazb\mathcal{R}xa$ and $czcv\mathcal{R}c$ so that $czc\mathcal{R}c$. Dually, $czc\mathcal{L}c$.

(2) $\Rightarrow$ (3) Assume that $czc\mathcal{H}c$. Then $c = czct$ for some $t \in S^1$. Thus $a = uc = uczct = azct$ and $a\mathcal{R}azc$. Also as $c\mathcal{L}a$ then by right congruence $czc\mathcal{L}azc$ and finally $a\mathcal{L}c\mathcal{L}czc\mathcal{L}azc$. Thus $a\mathcal{H}azc$.

(3) $\Rightarrow$ (2) Assume that $azc\mathcal{H}a$. Then $a = azct$ for some $t \in S^1$. Thus $c = xa = xazct = czct$ and $c\mathcal{R}czc$. Also as $c\mathcal{L}a$ then by right congruence $czc\mathcal{L}azc$ and finally $c\mathcal{L}a\mathcal{L}azc\mathcal{L}czc$. Thus $c\mathcal{H}czc$.

(2) $\Rightarrow$ (4) Assume that $czc\mathcal{H}c$. Then by Theorem 3.2.1 $(cz)^{\#}$ exists and $cz\mathcal{R}c\mathcal{R}b$. As $c\mathcal{L}a$ by right congruence $cz\mathcal{L}az$ and $H_{cz} = L_{cz} \cap R_{cz} = L_{az} \cap R_b$ is a group (equivalently contains an idempotent). Finally as $a = uc$ then by left congruence, $az = ucz\mathcal{R}uc = a$.

(4) $\Rightarrow$ (1) Assume that $az\mathcal{R}a$ and $(L_{az} \cap R_b)$ contains an idempotent e. As $az\mathcal{L}e$ and $e\mathcal{R}b$ then $aze = e$ and $eb = b$. It follows that $azb\mathcal{L}eb = b$ by right congruence and $az = aze\mathcal{R}azb$ by left congruence. Finally $a\mathcal{R}azb\mathcal{L}b$.

□

<u>Special cases:</u>

(1) Letting $z = 1$ is the classical theorem;

(2) Letting $a = b = d$, and $z = a$ in Theorem 5.2.1 we recover that $dad\mathcal{H}d$ iff $ad\mathcal{R}d$ and H_{ad} contains an idempotent (Theorem 3.2.1). Moreover, letting $c = a^{-d}$ we recover that if a is invertible along d then $dad\mathcal{H}d$ (since $c\mathcal{H}d$ and $cac = c$);

(3) Letting $z = a$ and $a = c$ we obtain existence criteria for the (b, c)-inverse;

(4) Letting $a = f$ and $b = e$ be idempotents, and $z = a$, we obtain that a is (e, f)-invertible iff $fac = facf$ is a unit in the local monoid fSf for some $c \in L_f \cap R_e$, iff $cae = ecae$ is a unit in the local monoid eSe for some $c \in L_f \cap R_e$;

(5) In particular, a is invertible along e iff $eae \in \mathrm{U}(eSe)$ (this is Lemma 4.3.1).

5.3) The categorical point of view and Reverse Order Laws

A very interesting feature of the (b, c)-inverse is that it can be understand as a genuine inverse of morphism, in a suitable category. This category is the *Schützenberger category* $\mathbb{D}(S)$ of the semigroup S, as defined by A. Costa and B. Steinberg in [43]*. It has for objects the elements of S, and morphisms are triples $f = (a, x, b)$ with $x \in aS^1 \cap S^1 b$. The domain of f is a, its codomain is b and we use the notation $f = a \xrightarrow{x} b$. If $x = au = vb$ and $g = (b, y, c) = b \xrightarrow{y} c$ is a morphism with $y = bw = rc$, then the composition is $g \circ f = a \xrightarrow{x} b \xrightarrow{y} c = a \xrightarrow{vy=xw} c$.

Among all the morphisms from b to c are the *trivial morphims*, of the form $f = c \xrightarrow{x=bac} c$. Next theorem claims that the (b, c)-inverses "are" the inverses of the trivial isomorphisms from b to c (hence the inverses along d are the inverses of the trivial isomorphisms in $Hom(d, d)$).

Theorem 5.3.1 ([154, Theorem 2.7]). Let S be a semigroup and $a, b, c \in S$. Then a is (b, c)-invertible iff $c \xrightarrow{cab} b$ is an isomorphism of $\mathbb{D}(S)$ ($cab \in R_c \cap L_b$), in which case its inverse morphism is $b \xrightarrow{a^{-(b,c)}} c$.

Not only does this theorem provide a graphical interpretation of the (b, c)-inverse (hence also of the inverse along an element), but it also opens the path to categorical proofs using composition properties. For instance, [154, Corollary 2.8] produces the equality

$$b \xrightarrow{a^{-(b,c)}} c = b \xrightarrow{b} cab \xrightarrow{c} c.$$

Also, we recover that $a^{-e} = (eae)^{-1}_{eSe}$, inverse of eae in the local submonoid eSe, and that $a^{-(e,f)}$ is the unique element $x \in eSf$ such that $x(fae) = e, (fae)x = f$.

This was put to a certain extent in [154] to study reverse order laws (can we compute the inverse of a product by using the product of the inverses?). We refer to [154] for the statements of the various ROLs therein. We only give one result here, to catch a glimpse of the type of results obtained.

Theorem 5.3.2 ([154, Theorem 2.7]). Let S be a semigroup and $a, w, b, s, t, c \in S$ be such that $a^{-(t,c)}$ and $w^{-(b,s)}$ exist. Then $(aw)^{-(b,c)}$ exists and equals $w^{-(b,s)}a^{-(t,c)}$ iff there exists $e \in E(S)$ such that:
(1) $t \xrightarrow{e} s$ is an invertible morphism;
(2) $caewb = cawb$.
In this case, st is a trace product (and e is the identity of the group $R_t \cap L_s$).

In case the equality $caewb = cawb$ does not hold but st is still a trace product with $e \in R_t \cap L_s$, then the ROL becomes $(aew)^{-(b,c)} = w^{-(b,s)}a^{-(t,c)}$ whenever $a^{-(t,c)}$ and $w^{-(b,s)}$ exist.

Chapter 6

Outer inverses, inverses along an element and partial orders

Partial orders on semigroups or more stringent structures (such as rings or matrices over fields) have a long history, and have been studied by many scholars. In [78] and [79] we compare with A. Guterman and P. Shteyner certain of these classical partial orders to new ones based on outer inverses (see also Chapter 18). In this chapter, I only present those results based on the inverse along an element. But before, we need some prerequisites on partial order on semigroups. More generic results on partial orders on arbitrary semigroups will be presented in Chapter 18.

6.1) The natural partial order on regular and arbitrary semigroups

The natural partial order on inverse semigroups was defined by Vagner in 1952 [205]*as an extension of the natural partial order on idempotents, and extended to the case of regular semigroups in 1980 independently by Hartwig [85]*and Nambooripad [178]*. Hartwig partial order (also called the minus partial order) is defined by $a <^- b \iff a'a = a'b$ and $aa' = ba'$ for some $a' \in V(a)$ (equiv. $a' \in I(a)$) and Nambooripad partial order is defined by: $a = eb$ and $aS \subseteq bS$ for some idempotent $e \in E(S)$ such that $a\mathcal{R}e$ ($aS^1 = eS^1$), or equivalently (for regular elements) $a = axb = bxa, a = axa$ for some $x \in S$, that we by denote $a <_{\mathcal{N}} b$. This order was later extended by Mitsch to non-regular semigroups [171]*: $a <_{\mathcal{M}} b \iff a = xb = by$, $xa = a$ for some $x, y \in S^1$. Mitsch partial order is sometimes also called the natural partial order, and denoted by $\leq$. Relations $<^-$ and $<_{\mathcal{N}}$ make sense on arbitrary semigroups, and still coincide in this case ([79, Proposition 1])*, but are then distinct from $<_{\mathcal{M}}$. Restricted to regular elements, all three relations coincide [171, Lemma 1]*and are also equivalent with $a = eb = bf$ for some $e, f \in E(S)$. Restricted to idempotents, they reduce to the natural partial order ($e \leq f \iff ef = fe = e$).

The minus partial order (and others) have particularly been studied in the context of matrices of the real or complex field [8]*, [76]*, [167]*, [169]*. Due to the the existing involution on the ring of real or complex matrices, it has then been compared with the *star order* and *dagger order* (also called Drazin partial order) [7]*, [89]*, where $A <^* B$ whenever $A^*A = A^*B$ and $AA^* = BA^*$ and $A <^\dagger B$ whenever $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$ ($A^\dagger$ is the Moore-Penrose inverse of A); if the Moore-Penrose exists for all elements, as it is the case for complex matrices, the two orders coincide (as seen for instance by cancellation properties). It has also been compared to the *sharp order* [168]*obtained by replacing the Moore-Penrose inverse with the group inverse ($A <^\# B$ whenever $A^\# A = A^\# B$ and $AA^\# = BA^\#$).

As in the non regular case relation $<^- = <_\mathcal{N}$ fails to be reflexive, we adopted in the papers [78] and [79] the convention that a partial order is an antisymmetric and transitive relation only. We do so in this section.

As one can see from the definitions, the partial orders make great use of reflexive inverses (or inner inverses). On the other hand, the study of partial orders based on outer inverses is less common [170]*, [30]*, [196]*.

6.2) Partial orders based on outer inverses

In [170]*, Mitra and Hartwig defined a relation $<^\Theta$ as follows. Let $\Theta : S \to \mathcal{P}(S)$ be a multi-valued function that sends an element to a subset of its outer inverses: $\Theta(a) \in W(a)$ for any $a \in S$. Then $a <^\Theta b$ if there exists some outer inverse $x \in \Theta(b)$ such that $a = bxb$.

They notably proved [170, Lemma 6]*that on regular semigroups, any partial order finer than the minus partial order is of this form for a specific choice of the function Θ. The two drawbacks of this definition are as follows:

- if $a <^{-\Theta} b$, then a is regular, whence the relation is not suitable for comparing nonregular elements;
- $<^{-\Theta}$ is not a partial order in general.

The main contribution of [78] is to define new relations based on outer inverses, some of them allowing to compare non-trivially non-regular elements and study some of their properties. In particular, by working with specific subsets of outer inverses defined by means of the inverse along an element, transitivity issues were considered. Below, I present these new relations, and some of their properties (notably the transitivity results). I also present the relation between the sharp partial order ($a <^\# b$ if $aa^\# = ba^\# = a^\# b$) , and partial orders based on inverses along elements in the (bi)commutant (centralizer and double centralizer).

Definition 6.2.1 ([78, Definitions 3.5 and 3.6]). Let Θ be a multi-valued function with values $\Theta(s) \subseteq W(s)$, for all $s \in S$. For any $a, b \in S$:
(1) $a <_{\mathcal{N}}^{\Theta} b$ if there exists $x \in \Theta(b)$ such that $a = axa = axb = bxa$;
(2) $a\Gamma^{\Theta}b$ if there exist $x \in \Theta(b)$ such that $a = axb = bya$ and $b\{1\} \subseteq a\{1\}$
(3) If b is not regular, then $a\Gamma_l^{\Theta}b$ (resp. $a\Gamma_r^{\Theta}b$, $a\Gamma_{\mathcal{P}}^{\Theta}b$) if there exists $x \in \Theta(b)$ such that $a = axb$ (resp. there exists $y \in \Theta(b)$ such that $a = bya$, there exist $x \in \Theta(b)$ such that $a = axb = bxa$);
(4) If b is regular, then $a\Gamma_l^{\Theta}b$ (resp. $a\Gamma_r^{\Theta}b$, $a\Gamma_{\mathcal{P}}^{\Theta}b$) if there exist $x, y \in \Theta(b)$, such that $a = axa = axb = bya$ (resp. there exist $x, y \in \Theta(b)$, such that $a = aya = axb = bya$, there exists $x \in \Theta(b)$, such that $a = axa = axb = bxa$).

It happens that $\Gamma^{\Theta} = \Gamma_l^{\Theta} \cap \Gamma_r^{\Theta}$ [78, Lemma 3.7], and that $<_{\mathcal{N}}^{\Theta} \subseteq \Gamma_{\mathcal{P}} \subseteq \Gamma$. Also, if $\Theta(s) = W(s)$, then $<_{\mathcal{N}}^{\Theta} = <_{\mathcal{N}} = <^{\Theta}$ (an it is a partial order).

6.3) The inverse along an element comes into play

We now make the connection with the inverse along an element. For any $\Delta : S \to P(S)$ we let $\Theta^{\Delta} : b \mapsto \{b^{-d} | d \in \Delta(b)\}$, that is, we only consider outer inverses of b of the form b^{-d} with $d \in \Delta(b)$. We make the following observation: any multi-valued map Θ is actually of this form, for $\Theta^{\Theta} = \Theta$ ([78, Lemma 3.10]). The associated relation $<^{\Theta^{\Delta}}$ will simply be denoted by $<^{-\Delta}$ (and similarly for $<_{\mathcal{N}}^{\Theta}$ etc...)

Recall that the starting point of our investigation of new relations based on outer inverses was the fact that $<^{\Theta}$ was not transitive (hence not a partial order) in general. Next two results investigate transitivity of $<_{\mathcal{N}}^{\Theta}, \Gamma^{\Theta}$ and $<^{-\Delta}$.

Proposition 6.3.1 ([78, Proposition 3.15]). For any Θ, $<_{\mathcal{N}}^{\Theta}, \Gamma_r^{\Theta}, \Gamma_l^{\Theta}, \Gamma^{\Theta}, \Gamma_{\mathcal{P}}^{\Theta}$ are partial orders.

Proposition 6.3.2 (from [78, Corollary 3.22 and Lemmas 3.23 and 3.24]). Let $\Delta : S \to \mathcal{P}(S)$. In the following cases, $<^{-\Delta}$ is a partial order:
(1) $\Delta : s \mapsto \{\delta_0\}$ is a constant singleton (in which case $a <^{-\Delta} b <^{-\Delta} c$ implies $a = b$);
(2) $\Delta : s \mapsto s$ (in which case $a <^{-\Delta} b$ iff $a = bb^{\#}b = b$);
(3) $\Delta : s \mapsto \{s^n, n \in \mathbb{N}\}$ (in which case $a <^{-\Delta} b$ iff $a = bb^D b$ and $a <^{-\Delta} b <^{-\Delta} c$ implies $a = b$);
(4) $\Delta(s)$ is a right or left ideal, for all $s \in S$;
(5) Δ is constant and a $\mathcal{R}, \mathcal{L}$ or $\mathcal{H}$-class.

Proposition 6.3.2 is actually more precise in the fourth case than the results in [78], where Δ is constant and the ideals are principal. However, the proof is essentially the same. We write it down below not only to be thorough, but also to show the reader how this partial order works precisely. Recall that $a <^{-\Delta} b$ if $a = bb^{-d}b$ for some $d \in \Delta(b)$.

Proof. Assume that $\Delta(s)$ is a right ideal, for all $s \in S$, and let $a, b, c \in S$ be such that $a <^{-\Delta} b <^{-\Delta} c$. Then $a = bb^{-d}b$ for some $d \in \Delta(b)$ and $b = cc^{-\delta}c$ for some $\delta \in \Delta(c)$.

Let $x = c^{-\delta}cb^{-d}cc^{-\delta}$. Then $cxc = a$ and $x = xcx$, so that $a = cc^{-x}c$. As $\Delta(c)$ is a right ideal, and as $x = c^{-\delta}cb^{-d}cc^{-\delta} = \delta\left((c\delta)^{\#}cb^{-d}cc^{-\delta}\right)$, and $\delta \in \Delta(c)$, then $x \in \Delta(c)$ (by the right ideal property). As also $x = a^{-x}$, this ends the proof. □

Two other interesting properties of the partial order $<_{\mathcal{N}}^{\Theta}$ are given by [78, Corollary 3.40] and [78, Lemma 3.47]:

(1) for any Θ, any invertible $a \in S$ is maximal with respect to $<_{\mathcal{N}}^{\Theta}$;

(2) $a <_{\mathcal{N}}^{\Theta} e$ $a \in S$, $e \in E(S)$ iff $a \in E(S)$ and $a \leq e$ (for the natural partial order).

Finally, we consider the relations $<^{-\Delta}$ and $<_{\mathcal{N}}^{-\Delta}$ for three specific functions Δ, and compare them to the sharp partial order $<^{\#}$ and the Drazin partial order $<^{\dagger}$:

- $\Delta = C : s \mapsto C(s) = \{s\}'$, commutant (or centralizer) of s;
- $\Delta = CC : s \mapsto CC(s) = \{s\}''$, bicommutant (or double centralizer) of s;
- $\Delta^* : s \mapsto \{x \in S | sx = (sx)^* \text{ and } xs = (xs)^*\}$ (in any $*$-semigroup).

Proposition 6.3.3 ([78, Corollaries 3.26 and 3.32 and Propositions 3.29 and 3.35])**.**

(1) $<^{-C}$, $<_{\mathcal{N}}^{-C}$, $<^{-CC}$ and $<_{\mathcal{N}}^{-CC}$ are partial orders;
(2) $<^{-CC} \subseteq <^{-C} = <^{\#} \subseteq <_{\mathcal{N}}^{-C}$ (and the two inclusions can be strict);
(3) For any $a, b \in S$, $a <^{-C} b$ iff $a <^{\#} b$ iff $a <^{-} b$ in the semigroup $C(b) = \{b\}'$;
(4) For any $a, b \in S$, $a <^{-CC} b$ iff $a <^{\#} b$ in the commutative semigroup $CC(b) = \{b\}''$ iff $a <^{-} b$ in $CC(b) = \{b\}''$.

Proposition 6.3.4 ([78, Proposition 3.37])**.** Let S be a $*$-semigroup. Then

$$<^{-\Delta^*} = <^{\dagger} \subseteq <_{\mathcal{N}}^{-\Delta^*}$$

(and the inclusion can be strict).

To conclude, let me add some words on the relations with other works.

- We have seen that in [77]*, Marki, Guterman and Shteyner introduce a general notion of quotient ring based on inverses along an element. In [77]*, they also compare some partial orders (notably the minus and the sharp partial orders) on the base ring and on the quotient ring.
- In [228]*, Zhu and Patricio consider partial orders based on the core and dual core inverse, which are special cases of inverses along an element.

Chapter 7

The ring case

In the ring case, all the previous results remain valid, but the additional sum operation brings both new methods and new questions. The ring theoretical methods used in my research are of four types, which are described below. I take advantage of the opportunity to express my deeepest gratitude to P. Patricio, who brought the first three methods to my attention, and from whom I learned a lot.

- Creation of units, and Jacobson's lemma;
- Use of matrix theory by embedding $R \hookrightarrow \mathcal{M}_2(R)$; in particular, use of the Schur complement;
- Use of matrix theory by using Peirce decompositions (or more generally Morita contexts) $R \simeq \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$, where $e \in E(R)$;
- use of module endomorphisms by identifying $R \simeq \mathrm{End}(R_R)$.

Another notion that is widely used in ring theory, notably as a replacement of principal ideals, is the notion of annihilator. The left annihilator of $a \in R$ is the left ideal $\{x \in R | xa = 0\}$. In the field of generalized inverses, this leads to the notions of *hybrid generalized inverses* and *annihilator generalized inverses* [51]*, [56]*, that allow to invert along non-regular elements (for regular elements, $aR = bR$ iff ${}_Rl(a) = Rl(b)$, so that these new generalized inverses are classical inverses along an element in this case).

Also, the ring case brings new questions. As we have seen with the reverse order law, questions about generalized inverses are usually instantiations of properties of the genuine inverse to other generalized inverses. More precisely, let $i : a \mapsto i(a)$ be a generalized inverse, and i_0 be the genuine inverse. The process is the following: if some formula $P(a, b, i_0)$ is valid, does it remains valid if one replaces the genuine inverse by a generalized inverse of a certain type? Otherly stated, is $P(a, b, i)$ valid? For instance, for the ROL $P(a, b, i_0) : (ab)^{-1} = b^{-1}a^{-1}$ whenever a and b are invertible, and for Cline's formula $P(a, b, i_0) : (ba)^{-1} = b(ab)^{-2}a$ whenever ab and ba are invertible (when applied to generalized inverses, one usually gets generalized invertibility of ba from that of ab for free). Passing from semigroups to rings allows to consider formulas involving sums. In my research, I have been considering this problem for Jacobson's lemma and the Absorption law:

- Jacobson's lemma - $P(a, b, i_0) : (1 - ba)^{-1} = b(1 - ab)^{-1}a$ whenever $(1 - ab)$ is invertible;
- Absorption law - $P(a, b, i_0) : a^{-1} + b^{-1} = a^{-1}(a + b)b^{-1}$ whenever a and b are invertible.

We will deal with these two questions in Sections 7.4 and 7.5.

7.1) Creation of units

It has long been known that strongly regular elements are related to certain units, precisely $a \in R$ is strongly regular iff $u = 1 + a - aa'$ is a unit for some (every) inner (reflexive) inverse of a. In that case, $a^{\#} = u^{-2}a$. Regarding inverses along an element, we proved with P. Patricio the following result. Recall that if a^{-d} exists, then d is regular.

Theorem 7.1.1 ([158, Theorem 3.2]). Let $a, d \in R$ with d regular and let d' be any inner (equiv. reflexive) inverse of d. Then the following statements are equivalent:
(1) a is invertible along d;
(2) $u = 1 + da - dd'$ is a unit;
(3) $v = 1 + ad - d'd$ is a unit.
In this case, $a^{-d} = u^{-1}d = dv^{-1}$.

For instance a is group invertible iff a^{-a} exists iff 1^{-a} exists iff $1 + a^2 - aa'$ is invertible for some (all) $a' \in I(a)$ iff $1 + a - aa'$ is invertible for some (all) $a' \in I(a)$.

If a is invertible along d, then $-a$ is also invertible along d (and a is also invertible along $-d$) since $-d\mathcal{H}d$. We obtain that a is invertible along d iff $-w = 1 - da - dd'$ is a unit in which case $a^{-d} = w^{-1}d$. This expresses da as a *clean element* (sum of an idempotent and a unit), $da = (1 - dd') + w$, an expression one can find in [227, Proposition 2.3]*. Moreover, this decomposition is *special clean* since $(1-dd')R \cap daR = 0$ (or equivalently, as we will see in part V, since $da = (da)w^{-1}(da)$), and satisfies the additional condition $(1 - dd')d = 0$.

In [153, Theorem 5.1] (see also Chapter 12), I relate strong regularity with cleanness as follows. An element $a \in R$ is strongly regular iff it is clean with clean decomposition $a = e + u$ ($e \in E(R), u \in R^{-1}$) such that $ae = 0$ (or dually such that $ea = 0$).

Combining the previous results we deduce the following result.

Theorem 7.1.2 (unpublished)**.** Let $a, d \in R$. Then the following statements are equivalent:
(1) a is invertible along d;
(2) da is clean with clean decomposition $da = e + u$ ($e \in E(R), u \in R^{-1}$), $eda = 0$ and $da\mathcal{R}d$;
(3) da is clean with clean decomposition $da = e + u$ and $ed = 0$;
In this case, the decomposition are special clean and $a^{-d} = u^{-1}d$.

As it is often the case, duality eases life. Here, dual statements (statements in the opposite ring) to (2) and (3) are also equivalent to (1) since this first statement is self-dual.

Proof. That $1 \Rightarrow 2$ follows from Theorem 3.2.1 and the previous arguments. The implication (2) $\Rightarrow$ (3) follows from cancellation.

First, we prove that the decomposition is special clean. So assume that da is clean with clean decomposition $da = e + u$, and that $eda = 0$ (let alone $ed = 0$). Let $x \in eR \cap daR$. Then $x = ex$ and $x = day$ for some $y \in R$. It follows that $x = eday = 0$ and the decomposition is special clean.

Second, we prove that (3) $\Rightarrow$ (1). As $ed = 0$ then $dad = ed+ud = ud$ and $(u^{-1}d)(ad) = d$. Also $0 = eda = ee+ew = e+ew$ so that $eu = -e = eu^{-1}$. Hence $dau^{-1} = eu^{-1}+1 = 1 - e$, and $dau^{-1}d = d$. Let $b = u^{-1}d$. Then $bad = d = dab$ and $b \leq_{\mathcal{L}} d$. It remains to prove that $b \leq_{\mathcal{R}} d$. We compute $dau^{-2}d = (1 - e)u^{-1}d = b + ed = b$. This ends the proof that a is invertible along d with inverse $a^{-1} = u^{-1}d$. □

As we have seen, the inverses along ((bi)commuting) idempotents play a special role in the theory. In the case of invertibility along an idempotent, the following result holds.

Corollary 7.1.3 ([147, Corollary 2])**.** Let $a \in R$ and $e \in E(R)$. Then the following statements are equivalent:
(1) a is invertible along e;
(2) $u = 1 + ea - e$ is a unit;
(3) $v = 1 + ae - e$ is a unit.
In this case, $a^{-e} = u^{-1}e = ev^{-1} = eu^{-1}e = ev^{-1}e$.

And finally, we present a result involving bijective centralizer.

Theorem 7.1.4 ([225, Theorem 3.7])**.** Let $a, d \in R$ with d regular and let d' be any inner (equiv. reflexive) inverse of d. Let also $\sigma : R \to R$ be a bijective centralizer. Then the following statements are equivalent:
(1) a is invertible along d;
(2) $u = 1 + \sigma(da) - dd'$ is a unit;
(3) $v = 1 + \sigma(ad) - d'd$ is a unit.
In this case, $a^{-d} = \sigma(u^{-1})d = d\sigma(v^{-1})$.

Note also that such characterizations of inverses along an element by means of units were also found for the one-sided inverse along an element [223, Corollaries 3.3 and

3.5]*, [37]*(u and v are then solely left and right invertible respectively).

7.2) Inverse along a triangular matrix

In [157], we study with P. Patricio the inverse of matrices (triangular or not) along triangular matrices over an arbitrary ring. The results therein exhibit four interesting global features (that appear naturally in any study on generalized inverses of matrices): Dedekind-finiteness of the ring R, use of units (as created in the previous section) and of zero products of the form $(1-e)x(1-f)$, where $e, f \in E(R)$, and use of Schur complement. Recall that a ring is *Dedekind-finite* if $(\forall a, b \in R)\ ab = 1 \Rightarrow ba = 1$. In the context of matrices, this is equivalent to saying invertible lower triangular matrices are exactly the matrices whose diagonal elements are ring units, and in this case the matrix inverse is again lower triangular.

Results of [157] are of two kinds; first we provide necessary and sufficient existence conditions, and second formulas for the inverse. And they are given first for triangular matrices A, and then arbitrary matrices A (but D is always assumed triangular).

Theorem 7.2.1 ([157, Theorem 2.2])**.** Suppose that R is Dedekind-finite and let $A = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ and $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_3 \end{bmatrix}$ be two matrices in $\mathcal{M}_2(R)$. Then A^{-D} exists iff a^{-d_1} and d^{-d_3} exist and $(1-d_3d_3^+)d_2(1-d_1^+d_1) = 0$ for some (all) $d_1^+ \in V(d_1), d_3^+ \in V(d_3)$. In this case,

$$A^{-D} = \begin{bmatrix} a^{-d_1} & 0 \\ v^{-1}d_2(1-d_1^+)a^{-d_1} + d^{\|d_3}(b + d_3^+d_2d_1^+)a^{\|d_1} + v^{-1}d_2 & d^{-d_3} \end{bmatrix},$$

with $v = d_3d + 1 - d_3d_3^-$ (for some (all) $d_3^- \in I(d_3)$).

In particular, by induction we obtain that given a Dedekind-finite regular ring R, if A^{-D} exists for two lower triangular matrices then all $a_{i,i}^{-d_{i,i}}$ exist and A^{-D} is again lower triangular [157, Theorem 2.3].

Without, Dedekind-finiteness, a similar result holds, but we need to assume that a^{-d_1} or d^{-d-2} exists (as just seen, the existence of A^{-D} in a Dedekind-finite ring implies the existence of both a^{-d_1} and d^{-d-2}).

Corollary 7.2.2 ([157, Corollary 3.2]). Let $A = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_3 \end{bmatrix}$ with D, d_1, d_3 regular, and suppose $a^{\|d_1}$ exists. Then A^{-D} exists iff $d^{\|d_3}$ exists and $(1 - d_3 d_3^+)d_2(1 - d_1^+ d_1) = 0$ for some (all) $d_1^+ \in V(d_1), d_3^+ \in V(d_3)$.
In this case, A^{-D} is lower triangular with

$$A^{-D} = \begin{bmatrix} a^{\|d_1} & 0 \\ v^{-1}d_2(1 - d_1^+)a^{\|d_1} + d^{\|d_3}(b + d_3^+ d_2 d_1^+)a^{\|d_1} + v^{-1}d_2 & d^{\|d_3} \end{bmatrix},$$

with $v = d_3 d + 1 - d_3 d_3^-$ (for some (all) $d_3^- \in I(d_3)$).

Corollary 7.2.3 ([157, Corollary 3.4]). Let $A = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_3 \end{bmatrix}$ with D, d_1, d_3 regular, and suppose $d^{\|d_3}$ exists. Then A^{-D} exists iff $a^{\|d_1}$ exists and $(1 - d_3 d_3^+)d_2(1 - d_1^+ d_1) = 0$ for some (all) $d_1^+ \in V(d_1), d_3^+ \in V(d_3)$.
In this case, A^{-D} is lower triangular, with

$$A^{-D} = \begin{bmatrix} a^{\|d_1} & 0 \\ v^{-1}d_2(1 - d_1^+)a^{\|d_1} + d^{\|d_3}(b + d_3^+ d_2 d_1^+)a^{\|d_1} + v^{-1}d_2 & d^{\|d_3} \end{bmatrix},$$

with $v = d_3 d + 1 - d_3 d_3^-$.

The previous two corollaries are actually special instances of next two theorems, where the matrix A is an arbitrary matrix (but D is still lower triangular). The results are however more intricate in these two general cases.

Theorem 7.2.4 ([157, Theorem 3.1]). Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_3 \end{bmatrix}$ be such that $a^{\|d_1}$ exists. Then A^{-D} exists iff

$$\zeta = \beta - \alpha a^{\|d_1} c$$

is a ring unit, where $d_1^+ \in V(d_1), d_3^+ \in V(d_3)$ and

$$\begin{aligned} w &= (1 - d_3 d_3^+)d_2(1 - d_1^+ d_1) \\ w^- &\in I(w) \\ \alpha &= d_2 a + d_3 b - (1 - ww^-)(1 - d_3 d_3^+)d_2 d_1^+ \\ \beta &= d_2 c + d_3 d + 1 - d_3 d_3^+ - ww^-(1 - d_3 d_3^+). \end{aligned}$$

In this case,

$$A^{-D} = \begin{bmatrix} a^{\|d_1} & -a^{\|d_1} c \zeta^{-1} d_3 \\ -\zeta^{-1}\alpha a^{\|d_1} + \zeta^{-1} d_2 & \zeta^{-1} d_3 \end{bmatrix}.$$

Theorem 7.2.5 ([157, Theorem 3.3]). Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_3 \end{bmatrix}$ be such that $d^{\|d_3}$ exists. Then A^{-D} exists iff

$$\xi = \gamma - cd^{\|d_3}\eta$$

is a ring unit, where $d_1^+ \in V(d_1), d_3^+ \in V(d_3)$ and

$$\begin{aligned} w &= (1 - d_3d_3^+)d_2(1 - d_1^+d_1) \\ w^- &\in I(w) \\ \gamma &= ad_1 + 1 - d_1^+d_1 - (1 - d_1^+d_1)w^-(1 - d_3d_3^+)d_2(1 - d_1^+d_1) \\ \eta &= bd_1 + dd_2 - d_3^+d_2(1 - d_1^+d_1) + d_3^+d_2(1 - d_1^+d_1)w^-(1 - d_3d_3^+)d_2(1 - d_1^+d_1) \end{aligned}$$

In this case,

$$A^{-D} = DV^{-1} = \begin{bmatrix} d_1\xi^{-1} & -d_1\xi^{-1}cd^{\|d_3} \\ d_2\xi^{-1} - d^{\|d_3}\eta\xi^{-1} & -d_2\xi^{-1}cd^{\|d_3} + d^{\|d_3}\eta\xi^{-1}cd^{\|d_3} + d^{\|d_3} \end{bmatrix}.$$

7.3) Natural inverse and generalized Drazin inverse

In the previous sections, we have defined (Definition 4.3.4) the $j-$maximal generalized inverses (resp. $j-$natural generalized inverse) of a $j = 0, 1, 2$, as a^{-M} for a maximal (resp. the greatest element) $M \in \Sigma_j(a)$ (if it exists). And we have seen that:

- if $\Sigma_2(a)$ is distributive, maximal implies natural (Proposition 4.3.5);
- the natural inverse generalizes the group and Drazin inverse (Theorem 4.3.6).

It happens that on any ring R, $\Sigma_2(a)$ is distributive semi-lattice (for any $a \in R$), so that we can focus only on natural inverses.

Using the additive operation, I proved in [147] that natural inverses have a "generalized" *core-nilpotent decomposition* (analogous to the *Jordan-Chevalley decomposition* that expresses linear operators as the sum of their commuting semi-simple part and their nilpotent part).

Theorem 7.3.1 ([147, Theorem 6]). Let $a \in R$. Then the following are equivalent:
1. a is naturally invertible with inverse a^{-M};
2. There exists $b \in \{a\}''$ such that $bab = b$ and $\Sigma_2(a - aba) = \{0\}$;
3. $a = x + y$ for some $x, y \in R$ such that $x \in \{a\}''$, $x^\#$ exists, $xy = 0 (= yx)$ and $\Sigma_2(y) = \{0\}$.

In this case, $a^{-M} = b = x^\#$.

The unique decomposition $a = x + y = aM + (a - aM) = aba + (a - aba)$ of this theorem is called the natural core decomposition of a.

Second, I make the link with Koliha-Drazin invertible (generalized Drazin invertible, quasi-polar) elements. First, let me recall some definitions. Let R be a ring.

- An element $q \in R$ is *quasinilpotent* if $\forall x \in \{q\}', 1 + xq \in R^{-1}$, and *quasi-quasinilpotent* if $\forall x \in \{q\}'', 1 + xq \in R^{-1}$;
- An element $a \in R$ is *quasipolar* (resp. *quasi-quasipolar*) if there exists a idempotent (called spectral idempotent) p in $\{a\}''$ such that ap is quasinilpotent (resp. quasi-quasinilpotent) and $a + p \in R^{-1}$;
- An element $a \in R$ is *generalized Drazin invertible* (resp. *quasi-generalized Drazin invertible*) if there exists b in $\{a\}''$ such that $bab = b$ and $a^2b - a$ is quasinilpotent (resp. quasi-quasinilpotent).

It was proved by J. Koliha and P. Patricio ([121, Theorem 4.2]*) that quasipolar elements are exactly the generalized Drazin invertible elements (also called Koliha-Drazin invertible elements).

Next theorem proves that the natural inverse generalizes not only the Drazin inverse, but also the Koliha-Drazin inverse in a ring.

Theorem 7.3.2 ([147, Theorem 8])**.** Let R be a unital ring, and $a \in R$ be (quasi-)quasipolar with spectral idempotent p and (quasi-)generalized Drazin invertible b. Then a is naturally invertible, $M = 1 - p$ is the greatest element of $\Sigma_2(a)$ and the (quasi-)generalized Drazin inverse b is equal to a^{-M}, the natural generalized inverse of a.

The converse statement does not appear in [147], but is straightforward. If a is naturally invertible with natural core decomposition $a = x + y$, then a is (quasi-)quasipolar iff y is (quasi-)quasinilpotent.

7.4) Jacobson's lemma

As Cline's formula (studied in Section 4.3), Jacobson's lemma is a property invariant by primarily conjugation, that relates invertibility of $1 - ab$ with that of $1 - ba$. It reads $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$.

In [151], we study Jacobson's lemma in the context of general (non-unital) rings, and then apply the results to the case of unital rings. In this section, since we present the results without proofs, we work the other way round. First, we present the (simpler) results in the classical unital case. Second, we give some insights on the general ring case (where it is a priori non-obvious what exactly could be Jacobson's lemma in lack of identity). As for Cline's formula, there will be one-sided and two-sided theorems.

Recall the following notations (for any $a \in R$):

$$\Sigma_0(a) = \{e \in E(S) | eae\mathcal{H}e\},$$
$$\Sigma_1(a) = \{a\}' \cap \Sigma_0(a),$$
$$\Sigma_2(a) = \{a\}'' \cap \Sigma_0(a),$$
$$\Sigma_R(a) = \{e \in E(S) | e \in aeS \cap Sae, eae = ae\},$$
$$\Sigma_L(a) = \{e \in E(S) | e \in eaS \cap Sea, eae = ea\}.$$

7.4.1) Jacobson's lemma in unital rings

- JACOBSON'S LEMMA FOR ONE-SIDED INVERSES IN UNITAL RINGS -

Corollary 7.4.1 ([151, Corollary 4.3]).
Let $e \in \Sigma_L(ab) \cap (1 - \Sigma_R(1 - ab))$. Then $f = b(ab)^{-e}a \in \Sigma_L(ba) \cap (1 - \Sigma_R(1 - ba))$, and

$$\begin{aligned}(1 - ba)^{-(1-f)} &= \left(1 + b\,(1 - ab)^{-(1-e)}\,a\right)\left(1 - b\,(ab)^{-e}\,a\right)\\ &= 1 + b\left((1 - ab)^{-(1-e)} - (1 - ab)^{-(1-e)}\,(ab)\,(ab)^{-e} - (ab)^{-e}\right)a.\end{aligned}$$

The assumption $e \in \Sigma_L(ab) \cap (1 - \Sigma_R(1 - ab))$ also reads $e \in \Sigma_L(ab)$ and $\bar{e} = (1 - e) \in \Sigma_R(1 - ab)$.

- JACOBSON'S LEMMA IN UNITAL RINGS -

Corollary 7.4.2 ([151, Corollary 4.5]).
Let $e \in \Sigma_j(ab) \cap (1 - \Sigma_j(1 - ab))$, $j = 1, 2$. Then $f = b(ab)^{-e}a \in \Sigma_j(ba) \cap (1 - \Sigma_j(1 - ba))$ and

$$\begin{aligned}(1 - ba)^{-(1-f)} &= 1 + b(1 - ab)^{-(1-e)}a - f\\ &= 1 + b\left((1 - ab)^{-(1-e)} - (ab)^{-e}\right)a.\end{aligned}$$

We have already seen that the spectral projection p of a generalized Drazin invertible element $1 - ab$ satisfies that $1 - p$ is the greatest element of $\Sigma_2(1 - ab)$ (Theorem 7.3.2 or [147, Theorem 8]). Actually, it is proved in [151, Example 4.2] that it also holds that $p \in \Sigma_2(ab)$. Thus $y = 1 + b\left((1 - ab)^{-(1-p)} - (ab)^{-p}\right)a$ seems a perfect candidate for the generalized Drazin inverse of ba. Actually, by the semillatices isomorphism properties, we already now that this is the natural inverse of $(1 - ba)$, and we have only to check that $(1 - ba)^2 y - (1 - ba)$ is quasinilpotent. This is done in [151, Example 4.3] and we recover Zhuang formula [229, Theorem 2.3]* (x^{gD} denotes the generalized Drazin inverse of $x \in R$).

$$(1 - ba)^{gD} = 1 + b\left((1 - ab)^{gD} - (ab)^{-p}\right)a,$$

where p is the spectral idempotent of $(1 - ab)$. Also, the spectral idempotent of $(1 - ba)$ is $q = b(ab)^{-p}a = b\left[p(1 - p(1 - ab))^{-1}\right]a$.

7.4.2) Jacobson's lemma in general rings

In order to deal with general rings, it has long been noticed that a interesting tool is the so-called *circle operation* $x \circ y = x + y - xy$. Indeed, it was first observed by Jacobson that this operation is associative, and that if R is a unital ring, then $x \mapsto 1 - x$ is an involutive isomorphism of monoids from $(R, .)$ onto $(R, \circ)$. (In some subsequent works on general rings, another operation has also been used, the *adjoint operation* $x * y = x + y + xy$; in this case $x \mapsto 1 + x$ is an involutive isomorphism of monoids from $(R, .)$ onto $(R, *)$.)

Let $\Re = (\Re, +, .)$ be a general ring. As 0 acts as an identity on $(\Re, \circ)$ then $(\Re, \circ)$ is a monoid, usually called the *adjoint semigroup with circle operation*, or *circle semigroup* of the general ring.

For practical reason, we will denote the circle semigroup as $\Re^\circ = (\Re, \circ)$. Let $a, b \in \Re$. By $\Sigma_R^\circ(a)$, ... we then mean the previous notions for the circle operation. If $e \circ a \circ e$ is invertible in $e \circ \Re \circ e$, then we will denote by $a^{\ominus e}$ this inverse, $a^{\ominus e} \circ (e \circ a \circ e) = e = (e \circ a \circ e) \circ a^{\ominus e}$(in unital rings, it then holds that $1 - a^{\ominus e} = (1-a)^{-(1-e)}$).

We can now state our versions of Jacobson's lemma in general rings.

- Jacobson's lemma for one-sided inverses in general rings -

Theorem 7.4.3 ([151, Theorem 4.1])**.**
Let $e \in \Sigma_L(ab) \cap \Sigma_R^\circ(ab)$. Pose $f = b(ab)^{-e}a$. Then $f \in \Sigma_L(ba) \cap \Sigma_R^\circ(ba)$ and

$$(ba)^{\ominus f} = \big(b(ab)^{\ominus e}a - ba\big) \circ f.$$

- Jacobson's lemma in general rings -

Theorem 7.4.4 ([151, Theorem 4.2])**.**
Let $e \in \Sigma_j(ab) \cap \Sigma_j^\circ(ab)$, $j = 1, 2$. Then $f = b(ab)^{-e}a \in \Sigma_j(ba) \cap \Sigma_j^\circ(ba)$ and

$$(ba)^{\ominus f} = \big(b\,(ab)^{\ominus e}\,a - ba\big) \circ \big(b\,(ab)^{-e}\,a\big) = b(ab)^{\ominus e}a - ba + b(ab)^{-e}a.$$

7.5) Absorption law

The absorption law claims that in a ring R, for any two invertible elements $a, b \in R^{-1}$, $a^{-1} + b^{-1} = a^{-1}(a+b)b^{-1}$. In [225], we prove (with H. Zhu, J. Chen and P. Patrício) that the absorption law is still valid for the (one-sided) inverses along a single element d.

Proposition 7.5.1 ([225, Proposition 2.2])**.** Let $a, b, d \in R$ such that a is left invertible along d and b is right invertible along d. Let also a_l^{-d} (resp. b_r^{-d}) be any left inverse of a (resp. right inverse of b) along d. Then

$$a_l^{-d} + b_r^{-d} = a_l^{-d}(a+b)b_r^{-d}.$$

In particular, for the (two-sided) inverse along d we obtain [225, Corollary 2.4]:

$$a^{-d} + b^{-d} = a^{-d}(a + b)b^{-d}.$$

Recall that, if $\sigma : R \to R$ is a bijective centralizer, then $d\mathcal{H}\sigma(d)$ [225, Proposition 3.5]. Thus we derive the following results.

Corollary 7.5.2 ([225, Theorem 2.7])**.** Let $\sigma : R \to R$ be a bijective centralizer and let $a, b, d \in R$ be such that a is invertible along $\sigma(d)$ and b is invertible along d. Then

$$a^{-\sigma(d)} + b^{-d} = a^{-\sigma(d)}(a + b)b^{-d}.$$

The interest lies in the study of specific inverses, such as the group, Drazin or Moore-Penrose inverse.

Corollary 7.5.3 ([225, Corollary 2.9])**.** Let $\sigma : R \to R$ be a bijective centralizer and let $a, b \in R$.
(1) if $a^{\#}, b^{\#}$ exist and $a = \sigma(b)$ then $a^{\#} + b^{\#} = a^{\#}(a + b)b^{\#}$;
(2) if a^D, b^D exist with the same index n and $a^n = \sigma(b^n)$ then $a^D + b^D = a^D(a + b)b^D$;
(3) (R is a ring with involution) if $a^{\dagger}, b^{\dagger}$ exist and $a^* = \sigma(b^*)$ then $a^{\dagger} + b^{\dagger} = a^{\dagger}(a+b)b^{\dagger}$;
(4) (R is a ring with involution) if $a^{\#}, b^{\dagger}$ exist and $a = \sigma(b^*)$ then $a^{\#} + b^{\dagger} = a^{\#}(a+b)b^{\dagger}$.

7.6) Miscellaneous of other results

In [11, theorem 5.1]*, the authors give the following formula, where R^{-d} denotes the set of elements of the ring R invertible along $d \in R$:

$$R^{-d} = d^{-}(dd^{-}Rdd^{-})^{-1} + (1 - d^{-}d)Rdd^{-} \oplus R(1 - dd^{-}).$$

In the particular case of inverses along an idempotent $e \in E(R)$, this reads:

$$R^{-e} = (eRe)^{-1} \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e)$$

and we recover the result of Lemma 4.3.1. In the same article, the authors also provide the reader with many representations of the inverse along an element in rings with involution.

To close this part, let me add a few words on applications to other settings, such as C^*-algebras or operators on Hilbert/Banach/Kreĭn spaces.

- In [144], I study the Moore-Penrose in Kreĭn spaces for both bounded and unbounded operators. Recall that a Kreĭn space $\mathcal{K}$ is a vector space endowed with an indefinite bilinear form of a certain kind (informally, $\mathcal{K}$ is a direct difference of two Hilbert spaces). As such, the algebra of bounded operators $\mathcal{B}(\mathcal{K})$ is not a C^*-algebra and classical results about the Moore-Penrose inverse do not hold. Theorems therein were then mostly of algebraic nature, leading to the study of generalized inverses in the simplest and most general setting possible, semigroups.

- In [147], the natural generalized inverse is also studied in Banach algebras, or for bounded operators on Banach spaces, using (local) spectral theory;
- [12]*expresses the inverse along an element as different kind of limits (series, integrals,...) in Banach and C^*-algebras, and more generally considers continuity issues.

Conclusion, open problems and future work

The inverse along an element, and its companion the (b, c)-inverse, have now reached a mature form, and are commonly used in works regarding generalized inverses. However, there are still interesting new properties to be discovered, even in the most general setting of semigroups, as the very recent article [210]*shows. I can see at least three (very broad) promising directions of future research regarding the inverse along an element, and future results will probably been obtained at the intersection of those roads:

(1) first, the study of the inverse along an element in some specific settings, such as C^*-algebras, or tensors, can surely be further developed;

(2) second, we should carry on the study of inverses along a specific element in such settings, for instance through spectral properties. I have particularly in mind inverses along (commuting) idempotents and the natural inverse;

(3) third, one should seek applications of the inverse along an element to study of other notions, such as partial orders, quotient rings (or semigroups of quotients), linear preservers,...

Regarding these three topics, I would be especially interested in the study of those elements (in a ring) such that a is invertible along e and $1 - a$ is invertible along $1 - e$ (as requested for Jacobson Lemma to hold). Once again, the special case of Banach or C^*-algebras may bring interesting results (in link with spectral properties). And if there is a greatest idempotent e such that a is invertible along e and $1 - a$ is invertible along $1 - e$ bicommuting with a, what would be the properties of such a *binatural inverse* in these cases? As we have seen, any a group, Drazin or generalized Drazin invertible is binaturally invertible in this sense.

It may also be interesting to study quotient rings/semigroups of quotients with respect to inverses along ((bi)-commuting) idempotents.

III

The group inverse

Part III - The group inverse

In this part, I collect different results I obtained along my research in link with a specific inverse, the *group inverse*. Chapter 8 provides the reader with new existence criteria for the group inverse in a semigroup. Then, Chapter 9 gives necessary and sufficient conditions for the existence of the group inverse of a product of regular elements in rings, under a simple extra assumption of regularity only. Formulas for the group inverse of such product are also provided. In Chapter 10, we consider the longstanding problem of the reverse order law for the group inverse: that is, when does the formula $(ab)^{\#} = b^{\#}a^{\#}$ hold. We solve the problem completely in stable semigroups and Dedekind-finite rings. In the general case, we only solve the two-sided reverse order law: $(ab)^{\#} = b^{\#}a^{\#}$ and $(ba)^{\#} = a^{\#}b^{\#}$. Chapter 11 considers an extension of unit-regularity by using group invertible elements of the semigroup instead of units (that may even not exist). Finally, we expose the link between special clean elements and group invertible elements of ring in Chapter 12 (special clean elements are the reflexive inverses of group elements).

Chapter 8

New existence criteria for the group inverse in a semigroup

8.1) Group inverse and units in a local subsemigroup

The following result proved very useful in the study of chains of associate idempotents. As it is stated under a different form, but relates to the group inverse, I state here the group inverse version (with a short proof -distinct from that in [156]- and based on Green's relations and Green's theorem [73, Theorem 7]*, see Theorem 2.1.1). This result was originally obtained in collaboration with P.P. Nielsen in our study of perspective rings, but appears actually implicitly in Miller and Clifford's article [166]*.

Proposition 8.1.1 (From [156, Proposition 2.2]). Let S be a semigroup and $a \in \mathrm{reg}(S)$ with $a\mathcal{R}e$, $e \in E(S)$. Then a is group invertible iff $eae \in \mathrm{U}(eSe)$.

Proof. Let b' be such that $ab' = e$ and let $b = b'ab'$. As $a\mathcal{R}e$ then $ea = ab'a = a$, so that $b' \in I(a)$ and $b \in V(a)$ with $ab = e$. Also, by Green's theorem a is group invertible iff $a\,\mathcal{H}\,a^2$, and $eae \in U(eSe)$ iff $eae\mathcal{H}e$ iff $a^2b\,\mathcal{H}\,ab$. So assume first that $a\,\mathcal{H}\,a^2$. By right congruence $ab\,\mathcal{L}\,a^2b$. And as $a\,\mathcal{R}\,a^2 = a^2ba$ then $a = a^2bax$ for some $x \in S^1$, so that $ab = a^2baxb$ and $ab\,\mathcal{R}\,a^2b$. Finally $ab\,\mathcal{H}\,a^2b$ or equivalently, $eae \in \mathrm{U}(eSe)$.
Conversely, assume that $eae \in \mathrm{U}(eSe)$. Then $ab\,\mathcal{H}\,a^2b$ and by right congruence, $a = aba\,\mathcal{L}\,a^2ba = a^2$. By left congruence, $b = bab\,\mathcal{R}\,ba^2b$. Thus $b = ba^2bx$ for some $x \in S^1$, and $a = aba = aba^2bxa = a^2xa$, so that $a\,\mathcal{R}\,a^2$. Finally $a\,\mathcal{H}\,a^2$. □

Consequently, using Lemma 4.3.1, a is group invertible iff it is invertible along some $e \in E(S) \cap R_a$, in which case $a^\# = (a^{-e})^2a$ and $a^{-e} = ea^\#e (= (ea)^\#e = e(ae)^\#)$.

Under the previous notations, letting $f = ba$ be the isomorphic idempotent, we recover [156, Proposition 2.2] thanks to Theorem 2.1.1 (since $a \in R_e \cap L_f$): $e = ab \sim_{r\ell} f = ba$ (equivalently, fe is a trace product by Theorem 1.1.1) iff $eae = a^2b \in \mathrm{U}(eSe)$. This actually happens to be also equivalent to $faf = ba^2 \in \mathrm{U}(fSf)$, by duality arguments (exposed below).

Proposition 8.1.2 (unpublished)**.** Let S be a semigroup and (a, b) be a regular pair. Let also $e = ab, f = ba$. Then $e = ab \sim_{r\ell} f = ba$ iff $eae = a^2b \in \mathrm{U}(eSe)$ iff $a\,\mathcal{H}\,a^2$ iff $faf = ba^2 \in \mathrm{U}(fSf)$.

Proof. Let $a \in S$. We have only to prove the equivalence $a\,\mathcal{H}\,a^2$ (a is group invertible) iff $faf = ba^2 \in \mathrm{U}(fSf)$. By duality (working in the opposite semigroup $S^{op} = (S, \times)$) in Proposition 8.1.1, a is group invertible in S^{op} iff $a \times a \times b \in \mathrm{U}(a \times b \times S \times a \times b)$ iff $faf = ba^2 \in \mathrm{U}(baSba) = \mathrm{U}(fSf)$. But a is group invertible in S^{op} iff it is group invertible in S, which concludes the proof. □

Finally, we obtain the following equivalences ((2) $\iff$ (4) is Theorem 1.1.2 and (4) $\iff$ (5) is Theorem 1.1.1):

Corollary 8.1.3 (unpublished)**.** Let S be a semigroup and (a, b) be a regular pair. Let also $e = ab, f = ba$. Then the following statements are equivalent:
(1) a is group invertible;
(2) $eae = a^2b$ is a unit in eSe;
(2′) $faf = ba^2$ is a unit in fSf;
(3) a is invertible along $e = ab$;
(3′) a is invertible along $f = ba$;
(4) $e = ab \sim_{r\ell} f = ba$;
(5) $fe = ba^2b \in R_f \cap L_e$ (fe is a trace product);
(6) $ba^2b\mathcal{H}b$.

We finally state a more general result that may be understood as a generalization of Miller and Clifford's theorem (the previous case is $z = 1$).

Proposition 8.1.4 (unpublished)**.** Let S be a semigroup, $e, f \in E(S)$ be isomorphic idempotents, $z \in S^1$ and $a \in R_e \cap L_f$. Then the following statements are equivalent:
(1) $fze \in R_f \cap L_e$;
(2) $eaze\mathcal{H}e$ ($eaze$ belongs to the group of units of eSe);
(2′) $fzaf\mathcal{H}f$.

Proof. As $a \in R_e \cap L_f$ then there exists $b \in R_f \cap L_e$ such that (a, b) is a regular pair with $ab = e, ba = f$.

(1) ⇒ (2) Assume that $fze \in R_f \cap L_e$.Then $baze\,\mathcal{R}\,ba$ and by left congruence $eaze = abaze\,\mathcal{R}\,aba = a\,\mathcal{R}\,e$. Second, as $fze = baze\,\mathcal{L}\,e$ then $eaze = aze\,\mathcal{L}\,e$. Finally $eaze\,\mathcal{H}\,e$.

(2) ⇒ (1) Assume that $eaze\,\mathcal{H}\,e$. As $ea = a = af$ then $afze = aze\,\mathcal{L}\,e$ and $fze\,\mathcal{L}\,e$. Second, as $aze = eaze\,\mathcal{R}\,e$ then by left congruence $baze\,\mathcal{R}\,bab = b\,\mathcal{R}\,f$.

□

(An even more general result is given in Theorem 5.2.1.)

8.2) Other characterizations of the group inverse in a semigroup

In [153, Theorem 5.1], I proved that in a ring R, $a = a^2x, x = x^2a$ for some $x \in V(a)$ iff $a \in R$ is group invertible iff $ae = a$, $e \in eaR \cap Rea$ for some idempotent $e \in E(R)$. While the proof of the first equivalence was done only using the multiplicative structure of the ring, the proof of the second one used additive decompositions. It happens that it remains true in the context of semigroups.

Corollary 8.2.1 (from [153, Theorem 5.1])**.** let S be a semigroup. Then $a = a^2x, x = x^2a$ for some $x \in V(a)$ iff $a \in S$ is group invertible iff $ae = a$, $e \in eaS \cap Sea$ for some idempotent $e \in E(S)$.

Proof. Assume that $ae = a$, $e \in eaS \cap Sea$ for some idempotent $e \in E(S)$, and let $x, y \in S$ be such that $e = eax = yea$. Since $ae = a$ then $e = (eae)(exe) = (eye)(eae)$ and eae is both left and right invertible hence invertible in eSe. Also, $e \mathcal{L} a$ since $ae = a$ and $e = yea$. By the dual of Proposition 8.1.1 a is group invertible.
Conversely, assume that a is group invertible. Then $e = aa^{\#}$ satisfies that $ae = a$, $e \in eaS \cap Sea$. □

After publication of the article [153], I actually found that first the equivalence had already been proved in the context of semigroups by M. Petrich [191, Lemma 3.3]*. This also relates to the following global statement [45, Theorem 2]*(see also [194, Theorem IV. 1.6]*or [97, Theorem 2]*): A regular semigroup S that is also left regular ($a \in Sa^2$ for any $a \in S$) is completely regular.

Chapter 9

The group inverse of a product in rings

Let $a, b \in R$ be any two regular elements of a ring R. In this section, we follow [159] and find necessary and sufficient conditions for the existence of the group inverse of the product ab under a simple extra regularity condition. Moreover we obtain a formula for this inverse.

I will first present the results P. Patricio and myself obtained in [159], and then dive more precisely into the methodology of the paper (that completely disappears in the statements of the results) to understand where the various quantities come from and what mathematical arguments hide behind the results.

Throughout this chapter, R is unital ring.

9.1) Existence and characterization of $(ab)^{\#}$

Theorem 9.1.1 ([159, Theorem 2.2])**.** Let a, b be regular elements in R with reflexive inverses a^+ and b^+, respectively. Assume, also, that $w = (1 - bb^+)(1 - a^+a)$ is regular. Then $(ab)^{\#}$ exists if and only if $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$ is a unit for some inner inverse w^- of w. In this case,

$$(ab)^{\#} = az^{-2}b.$$

By duality, we deduce the following corollary.

Corollary 9.1.2. Let a, b be regular elements in R with reflexive inverses a^+ and b^+, respectively. Assume, also, that $w = (1 - bb^+)(1 - a^+a)$ is regular. Then $(ab)^{\#}$ exists if and only if $t = 1 - bb^+ + ba + (1 - ww^-)(1 - bb^+)$ is a unit for some inner inverse w^- of w. In this case,

$$(ab)^{\#} = at^{-2}b.$$

It is actually proved in [159] that such formulas extend to positive and negative powers of ab (precisely, the subgroups generated by z (equiv. t) and ab - G_z and G_{ab} - are isomorphic via the isomorphism $\phi : z^k \mapsto az^{k-1}b$). This gives the formula

$$\boxed{(\forall n \in \mathbb{Z})\ (ab)^{n+1} = az^n b = at^n b.}$$

For $n = -2$ we recover that $(ab)^\# = az^{-2}b = at^{-2}b$, and for $n = -1$ we obtain the identity $e = az^{-1}b = at^{-1}b = (ab)(ab)^\#$ of the group.

9.2) The methodology

Perhaps surprisingly, it happens that the main tool of the paper is matrix theory. Consider the matrix $M = \begin{bmatrix} ab & a \\ 0 & 1 \end{bmatrix} = AQ$ with $A = \begin{bmatrix} a & 0 \\ 1 & -b \end{bmatrix}, Q = \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix}$.

It is known that $M^\#$ exists iff $(ab)^\#$ exists (see for instance Corollary 7.2.3 or Proposition 8.1.1). Furthermore, the (1,1) entry of $M^\#$ equals $(ab)^\#$. Thus we have to compute this entry.

As Q is invertible then the group inverse of $M = AQ$ exists iff $U = AQ + I - (AQ)(Q^{-1}A^-)$ is a unit for some (all) inner inverse A^- of A (1 is invertible along $M = AQ$ and creation of units - Theorem 7.1.1), in which case $M^\# = U^{-2}M$. By carefully choosing A^-, the expression of $G = UK$ becomes tractable for some invertible lower triangular matrix K. It remains to characterize when $G = UK = \begin{bmatrix} 1 & a \\ \alpha & 2 - bb^+ - ww^-(1 - bb^+) \end{bmatrix}$, is invertible, where

$$\begin{aligned} \alpha &= (1 - ww^-)(1 - bb^+)a^+ + \big(2 - bb^+ - ww^-(1 - bb^+)\big)\,(a^+ - b) \\ &= a^+ - b + 2(1 - ww^-)(1 - bb^+)a^+, \end{aligned}$$

and compute its inverse. This can be done by using *Schur complement* of the (1,1)-entry, a very useful replacement of the determinant criterion of linear algebra valid for matrices over arbitrary rings, and more generally *Morita contexts* (see [159, Lemma 2.1] for a precise statement of the criterion). But this Schur complement is precisely $G/I = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$, giving the existence criterion. Then we compute the inverse of G, and finally obtain the (1,1)-entry of $M^\#$.

Chapter 10

The "Reverse Order Law" for the group inverse in semigroups and rings

In this section, we continue our study of group inverses of products, but we now also assume that a and b are group invertible. Our ultimate goal is to provide necessary and sufficient conditions for the one-sided reverse order law (ROL) to hold:

$$(ab)^{\#} = b^{\#}a^{\#}.$$

To achieve this goal, we first study the two-sided ROL for the group inverse:

$$(ab)^{\#} = b^{\#}a^{\#} \text{ and } (ba)^{\#} = a^{\#}b^{\#},$$

in semigroups and rings. Second, we prove that **under finiteness conditions**, the two sided ROL is actually equivalent with the one-sided ROL.

10.1) The two-sided ROL and $\mathcal{H}$-commutation

The main contribution of [149] is to relate the ROL for the group inverse to Green's preorders. In the sequel, $K_{\leq a} = \{x \in S | x \leq_{\mathcal{K}} a\}$ for $\mathcal{K} = \mathcal{L}, \mathcal{R}, \mathcal{H}$.

First, certain "inequalities" imply the two-sided ROL.

Lemma 10.1.1 ([149, Lemma 2.2])**.** Let $a, b \in S$ be group elements such that $ab \in L_{\leq a} \cap R_{\leq b}$ and $ba \in L_{\leq b} \cap R_{\leq a}$. Then ab and ba are group invertible and $(ab)^{\#} = b^{\#}a^{\#}$, $(ba)^{\#} = a^{\#}b^{\#}$.

Second, the equality $(ba)^{\#} = a^{\#}b^{\#}$ relates to Green's preorder $\leq_{\mathcal{H}}$.

Lemma 10.1.2 ([149, Lemma 2.3])**.** Let S be a semigroup and $a, b \in S$ be group elements such that ab is group invertible and $(ab)^{\#} = b^{\#}a^{\#}$. Then $ab \leq_{\mathcal{H}} ba$.

Combining these lemmas, and using knowledge on $\mathcal{H}$-commutation as studied in [2]*, we derive the main theorem of [149].

Theorem 10.1.3 ([149, Theorem 2.4]). Let S be a semigroup and $a, b \in S$ be group elements. Let $a^0 = aa^\#, b^0 = bb^\#$. Then the following statements are equivalent:
(1) ab and ba are group invertible with $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$;
(2) $ab\mathcal{H}ba$;
(3) $(\exists x, y \in S^1)$ $ab = bxa$ and $ba = ayb$ ($ab \in bS^1a$ and $ba \in aS^1b$);
(4) $ab \in L_{\leq a} \cap R_{\leq b}$ and $ba \in L_{\leq b} \cap R_{\leq a}$;
(5) $ab, ba \in H_{\leq a} \cap H_{\leq b}$;
(6) $a^0 \in \{b\}'$ and $b^0 \in \{a\}'$;
(7) $a^0, b^0 \in \{a, a^\#, a^0, b, b^\#, b^0\}'$;
(8) The subsemigroup C of S generated by $\{a, a^\#, b, b^\#\}$ is a Clifford semigroup.

A global version of this element-wise theorem follows.

Theorem 10.1.4 ([149, Theorem 2.7]). Let S be semigroup. Then the following statements are equivalent:
(1) S is completely regular and $(\forall a, b \in S)\, (ab)^\# = b^\# a^\#$;
(2) S is regular and $(\forall a, b \in S)\, ab\mathcal{H}ba$;
(3) S is regular and $(\forall a, b \in S)\, ab \in L_{\leq a} \cap R_{\leq b}$;
(4) S is a Clifford semigroup.

10.2) The one-sided ROL under finiteness conditions

[149, Example 3.1] (in semigroups) and [149, Example 3.9] (in rings) show that in general, the one-sided ROL does not imply the two-sided one (equiv. $ab\mathcal{H}ba$). However, I was able to prove that **this is the case under local [149, Theorem 3.3] or global finiteness conditions [149, Theorem 3.10], [149, Theorem 3.16].**

These conditions are:
(1) Drazin index $i(ba) \leq 1$; that is ba is group invertible [149, Theorem 3.3];
(2) Minimal condition on principal left ideals $\mathcal{M}_L$: every set of principal left (resp. right) ideals of S contains a minimal member with respect to inclusion;
(3) Left stability: S is left stable if $(\forall a, b \in S)\; ab\,\mathcal{J}\,b \Rightarrow ab\,\mathcal{L}\,b$ [149, Theorem 3.10];
(4) Dedekind-finiteness for rings ($ab = 1 \Rightarrow ba = 1$) [149, Theorem 3.16].

We conclude with some comments.

- The proof of [149, Theorem 3.16] uses Peirce matrix rings. Precisely, for a group invertible and $e = aa^\#$, we decompose the ring R as

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

(in matrix form $R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$). Then we embed R in $\mathcal{M}_2(R)$ and use that in Dedekind-finite rings, group inverses of upper triangular matrices are upper triangular;

- Obviously, the dual conditions $\mathcal{M}_R$ and right stability also work;
- By [41, Lemma 6.41]*left stability is equivalent to Munn's condition $\mathcal{M}_L^*$ [176]*, a weaker condition than $\mathcal{M}_L$. For a modern presentation of the topic, see [61]*;
- The minimal condition $\mathcal{M}_L$ is equivalent to *left DCCP*: every strictly descending chain of principal left ideals of S breaks off after a finite number of terms. In case of a ring R, this is also equivalent to the ring being *right perfect* (a ring R is right perfect if left R-modules have a projective cover). Such rings are automatically Dedekind-finite;
- Recently, it has been proved that regular Dedekind-finite ring are completely semisimple [123]*. It follows that regular Dedekind-finite rings are (left and right) stable.

Chapter 11

Group-regular and group-dominated elements in semigroups and rings

11.1) Position of the problem

Among the various specializations of regularity, unit-regularity ($aua = a$ for some unit u) plays a prominent role. This is especially the case in the context of rings where *unit-regular rings* are largely studied [24]*, [26]*, [32]*, [34]*, [62]*, [88]*, notably in link with internal cancellation [82]*, [110]*, [138]*. They are also studied on the level of elements [86]*, [131]*, recently in link with the clean property [109]*, [185]*, [216]*. Unit-regularity is also studied (but to a lesser extend) in the monoid case. Unit-regular inverse or orthodox monoids are for instance studied in [15]*, [17]*, [163]*, [164]*or [201]*whereas [21]*, [99]*or [195]*deal with the case of algebraic monoids. In the general case, we can cite [16]*, [46]*, [94]*or [202]*. Unit-regular monoids are also sometimes termed *factorisable monoids*. Indeed, it is well known that unit-regular elements of a monoid $\mathcal{M}$ can be characterized as elements of the form $a = eu$ (resp. $a = ue$) with $e \in E(\mathcal{M})$ and $u \in \mathcal{M}^{-1}$. In particular, a unit-regular monoid $\mathcal{M}$ satisfies $\mathcal{M} = E(\mathcal{M})H_1 = H_1E(\mathcal{M})$, where $H_1 = \mathcal{M}^{-1}$ is the group of units of the monoid (see [46]*for a more general definition of factorisable semigroups, and the equivalence between the two notions for monoids).

The aim of this section is to expose the development of a concept close to unit-regularity using maximal subgroups of a semigroup instead of solely the group of units. This idea appears (in a different context) notably in the work of Fountain, Petrich, Gould and others on orders on semigroup (equivalently semigroups of quotients) [67]*, [71]*. As noted by V. Gould in [72]*: *“Their aim was to develop concepts that reflect the equal importance of all subgroups of a semigroup, not only the group of units, which of course may not even exist.”*.

While it is then tempting to replace directly the group of units by the union of the maximal subgroups $H(S) = S^{\#}$, that is replace units by local units (group invertible elements) (this is done in [160] in the context of rings), doing uniquely this may not be sufficient in the context of semigroups, notably to get structure theorems. Indeed, a crucial property of units in a monoid is that they are majorants for the preorder $\leq_{\mathcal{H}}$ and maximal for the $\leq$ preorder (Mitch's extension of the natural partial order on idempotents [171]*). Indeed, for any $a \in S$ and $u \in S^{-1}$, $a = au^{-1}u = uu^{-1}a$ and a$\leq_{\mathcal{H}} u$. Also, if $u \leq a$, $u \in S^{-1}$, then $\exists e, f \in E(S), u = ea = af$. It follows that $eu = u$ hence $e = euu^{-1} = 1$, $a = u$ and $f = 1$. As second feature is that the identity is a central idempotent. Recall that a regular semigroup with central (resp. commuting) idempotents is a Clifford (resp. inverse) semigroup.

In my work, I proposed the following definitions. Recall (Chapter 6) that for any two elements $a, b \in S$, $a \leq b$ iff $a = xb = by$, $xa = a$ for some $x, y \in S^1$, and that for regular elements this is equivalent to $a = eb = bf$ for some $e, f \in E(S)$.

Definition 11.1.1 ([150, Definition 6.1])**.** Let S be a semigroup, T a subset of S. An element $a \in S$ is *T-regular* (resp. *T-dominated*) if it admits an inner inverse (resp. a majorant for the natural partial order) $x \in T$. S is T-regular (resp. T-dominated) if each element is T-regular (resp. T-dominated).

In the particular case $T = S^{\#}$ is the set of group invertible elements, this leads to the following instantiation (in general, we will only assume that $T \subseteq S^{\#}$ and $T = T^{\#}$).

Definition 11.1.2 ([160, Definition 2.1])**.** Let S be a semigroup and $a \in S$. We say that:
(1) a is group-regular if $a = axa$ for some $x \in S^{\#}$;
(2) a is intra group-regular if there exists $x \in S^{\#}$ such that $axa = a$ and $a^2 = axx^{\#}a$;
(3) a is group-dominated if $a \leq x^{\#}$ for some $x \in S^{\#}$.
S is group-regular (resp. intra group-regular, group-dominated) if every element of S is group-regular.

11.2) General results in the semigroup case

Next lemma presents some link between the previous notions in the most general context of semigroups.

Lemma 11.2.1 ([150, Lemma 6.2], [160, Proposition 2.2 and Corollary 2.3])**.** Let $a \in S$, $x \in S^{\#}$. Then the following statements are equivalent:
(1) $a \leq x^{\#}$;
(2) $a \leq_{\mathcal{H}} x^{\#}$ and $axa = a$;
(3) $a \leq_{\mathcal{H}} x$ and $axa = a$;
(4) a is unit-regular in the local submonoid $xx^{\#}Sxx^{\#}$ with inverse x;
(5) $a = ex^{\#}$ for some $e \in E(S)$ such that $e \leq xx^{\#}$;
(6) $a = x^{\#}f$ for some $f \in E(S)$ such that $f \leq xx^{\#}$.
In particular, group-dominated implies intra group-regular implies group-regular.

The equivalence between (1) and (4) states that group-domination is equivalent to a localized version of unit-regularity (a is group-dominated iff it is locally unit-regular), while (1) $\iff$ (5) $\iff$ (6) implies the following factorization property: if S is group-dominated then $S = E(S)S^\# = S^\# E(S)$. More generally, if S is T-dominated for some $T \subseteq S^\#$ such that $T^\# = T$ then $S = E(S)T = TE(S)$.

Also, we may note that if $a = axa$ for some $x \in S^\#$ such that $xx^\# \in Z(S)$ ($xx^\#$ is central) then $a \leq x^\#$.

11.3) The case of completely E-simple, completely (E, H_E)-abundant and E-Clifford restriction semigroups

In [150], I studied the structure of completely E-simple, completely (E, H_E)-abundant and E-Clifford restriction semigroups (where E is a specific subset of idempotents) which are also T-dominated or T-regular, for $T = \bigcup_{e \in E} H_e$. The results are highly technical and need too much definitions to be exposed in this section (for the reader interested in the details, I refer to Chapter 15 or directly to [150]). However, we may summarize the results therein as follows:

- In this specific case, T-regularity and T-domination are equivalent concepts;
- The structure of such semigroups is well-known, and based on factorisable (a.k.a. unit-regular) monoids (rather than mere monoids in the general case);
- In case E is central, T is a Clifford semigroup and the semigroup S is a strong semilattice of factorisable monoids; The converse also holds.

11.4) The ring case

In the ring case, many simplifications occur. We first consider the case of a unital ring. In this specific case, it happens that group-regularity boils down to unit-regularity.

Corollary 11.4.1 ([160, Corollary 2.5])**.** Let $a \in R$ unital ring. Then the following statements are equivalent:
(1) a is unit-regular;
(2) a is group-dominated (locally unit-regular);
(3) a is intra group-regular;
(4) a is group-regular.

In fact, as observed by Professor T.Y. Lam, more is true. T.Y. Lam and D. Khurana proved some years ago (private communication, to appear in [126]*, see also [113, Theorem 2.17]*) that an element $a \in R$ is unit-regular iff it has a unit-regular inner inverse. We can also deduce it from the previous corollary and the well-known fact that a regular product of two idempotents admits a idempotent reflexive inverse [65]*.

Corollary 11.4.2 (unpublished)**.** Let $a \in R$ unital ring. Then the following statements are equivalent:
(1) a is unit-regular;
(2) a has a unit-regular reflexive inverse;
(3) a has a unit-regular inner inverse.

Proof.

(1) ⇒ (2) Let $a \in R$ be unit-regular with unit-inverse $u^{-1} \in \mathrm{U}(R)$. Then $b = u^{-1}au^{-1} \in V(a)$ and $bub = u^{-1}au^{-1}uu^{-1}au^{-1} = u^{-1}(au^{-1}a)u^{-1} = u^{-1}au^{-1} = b$, so that b is unit-regular.

(2) ⇒ (3) this is a tautology.

(1) ⇒ (2) Let $a \in R$ have a unit-regular inner inverse b, with $bu^{-1}b = b$ for some $u \in \mathrm{U}(R)$. Then $au = (ab)(u^{-1}bau)$ with $ab, ba \in E(R)$. By similarity, $u^{-1}bau$ is also idempotent, and au is a product of idempotents. But au is regular (with inner inverse $u^{-1}b$). Let $e = (u^{-1}bau)(u^{-1}b)(ab) = u^{-1}bab$. As $aue = ab$ then $aueau = au$ ($e \in I(au)$) and $e^2 = u^{-1}babu^{-1}bab = u^{-1}bab = e$ so that $e \in E(R)$. Finally, au has an idempotent inner inverse. As idempotent are group invertible, then au is group-regular hence unit-regular by Corollary 11.4.1. Let $v^{-1} \in \mathrm{U}(R)$ be a unit-inverse of au. Then $a = auu^{-1} = auv^{-1}auu^{-1} = a(uv^{-1})a$ and a is unit-regular.

□

This exhibits the set of unit-regular elements of a ring as a fixed point of the map $V : \mathcal{P}(R) \to \mathcal{P}(R)$ (resp. $I : \mathcal{P}(R) \to \mathcal{P}(R)$) that maps any subset $X \subseteq R$ to its set of reflexive (resp. inner) inverses.

Also (and anticipating slightly on the results of next section) we can strengthen the previous corollaries in the case of unit-regular *rings* to the following one (proved independently by Khurana et.al. [113, Theorem 2.17]*).

Corollary 11.4.3 ([153, Corollary 4.2])**.** Let R be a unital ring. The the following statements are equivalent:
(1) R is unit-regular;
(2) Any element of R has an inner inverse that is group invertible (R is group-regular);
(3) Any element of R has a reflexive inverse that is group invertible (R is special clean).

This result is not valid element-wise for there exist unit-regular elements with no group invertible reflexive inverse.

However, **in lack of identity**, a group-regular element needs not be group-dominated, as proven by [160, Example 2.4].

Surprisingly, while distinct element-wise, the two concepts (of group-domination and group-regularity) become equivalent if considered globally, even in **non-unital rings**. That is group-regular (general) rings and group-dominated (general) rings are the same.

This is based on the following observation. If an element is group-regular in a corner ring, then it is actually unit-regular in this corner ring hence group-dominated in the whole ring. This happens for instance if any finite set of elements of the ring lies in a corner ring. Such rings are usually called *rings with "local units"* ([3, Definition 1]*), but the terminology may however have other meanings. As Von Neumann regular general rings have local units we obtain the following theorem.

Theorem 11.4.4. Let $\Re$ be a general ring. Then the following statements are equivalent:
(1) $\Re$ is group-dominated ($\forall a \in \Re$, a is group-dominated);
(2) $\Re$ is group-regular ($\forall a \in \Re$, a is group-regular).

Obviously, this is also equivalent with being an intra-group-regular ring by Lemma 11.2.1. A closer study of group-regular rings will be pursued in Section 22.1.

Chapter 12

Group invertible and (special) clean elements (in rings)

12.1) A new characterization of strongly regular elements via clean decompositions

Let R be a ring. It is well known that strongly regular but also strongly π-regular elements are *strongly clean*, where $a \in R$ is strongly clean iff $a = \bar{e} + u$ for some $e \in E(R)$ and $u \in R^{-1}$ such that $ae = ea$. In [153], I proved the following equivalences.

Theorem 12.1.1 ([153, Theorem 5.1])**.** Let $a \in R$. then the following statements are equivalent:
(1) $ae = a$, $e \in eaR \cap Rea$ for some idempotent $e \in E(R)$;
(2) $a = \bar{e} + u, a\bar{e} = 0$ for some $e \in E(R), u \in R^{-1}$;
(3) There exists $x \in R$ such that $axa = a = a^2x, xax = x = x^2a$;
(4) a is strongly regular (a.k.a. group invertible);
(5) $a = \bar{e} + u, a\bar{e} = \bar{e}a = 0$ for some $e \in E(R), u \in R^{-1}$.

That (1) $\iff$ (3) $\iff$ (4) also holds in semigroups was discussed in Section 8.2. The equivalence (4) $\iff$ (5) was already known [47, Proposition 2.5]*, see also [84]*. The implication (2) $\Rightarrow$ (5) claims that in the definition of simple polarity (that is equivalent with (5)), the commutativity assumption is not needed. One must however be cautious with this implication. Indeed, we did not claim that $a = e + u$ for some $e \in E(R), u \in R^{-1}$ such that $a\bar{e} = 0$ implies $\bar{e}a = 0$, but only that there exists a second idempotent f and a second unit v such that $a = \bar{f} + v$ with $a\bar{f} = \bar{f}a = 0$. Since the spectral idempotent in strongly regular decompositions is unique ([121, Proposition 2.6]*or [47, Proposition 2.6]*), this second idempotent is $f = ueu^{-1}$. And $f \neq e$ unless $eu = ue$. A direct and more visual proof of the implication (2) $\Rightarrow$ (5) can be done using the Peirce decomposition $R = eRe \oplus eR\bar{e} \oplus \bar{e}Re \oplus \bar{e}R\bar{e}$.

12.2) Characterization of special clean elements by group invertible ones

Recall that an element $a \in R$ is *special clean* (see [1]*, [26]*) if it admits a clean decomposition $a = \bar{e} + u$ for some $e \in E(R)$, $u \in \mathrm{U}(R)$ that satisfies the additional requirement $aR \cap \bar{e}R = \{0\}$.

These special clean elements appear in almost all my publications in the realm of ring theory, for the following reason: they can be described entirely multiplicatively by means of strongly regular elements. Precisely, I proved the following equivalences.

Theorem 12.2.1 ([153, Theorem 4.1], [141, Lemma 2.2 and Theorem 2.4], [139, Proposition 4.20])**.** Let R be a ring and $a \in R, e \in E(R)$. The following statements are equivalent:
(1) $u = a - \bar{e} \in \mathrm{U}(R)$ and $aR \cap \bar{e}R = 0$ (a is special clean);
(1′) $u = a - \bar{e} \in U(R)$ and $Ra \cap R\bar{e} = 0$;
(2) $u = a - \bar{e} \in U(R)$ and $aR \oplus \bar{e}R = R$;
(2′) $u = a - \bar{e} \in U(R)$ and $Ra \oplus R\bar{e} = R$;
(3) $aR \oplus \bar{e}R = R$ and $Ra \oplus R\bar{e} = R$;
(4) $aR \oplus \bar{e}R = R$ and $bR \oplus \bar{e}R = R$, for some $b \in V(a)$;
(5) $u = a - \bar{e} \in U(R)$ and $a = au^{-1}a$;
(6) $u = a - \bar{e} \in U(R)$, $z = u^{-1}au^{-1} \in V(a) \cap R^{\#}$ (z is a reflexive inverse of a which is strongly regular) and $zz^{\#} = e$;
(7) $aza = a, zaz = z$ and $zz^{\#} = e$ for some $z \in R^{\#}$.

Observe that the equivalence (1) $\iff$ (3) (for instance) proves the left-right symmetry of the concept of special clean element. While direct sums of right modules have been extensively studied, mixed-type decompositions (involving both right and left modules) have attracted less attention. Condition (3) claims that the direct sums conditions in (2) and (2′) together actually imply invertibility of u (hence that a is special clean). The equivalence (1) $\iff$ (4) claims that a is special clean iff aR and bR are *perspective* (share a common complementary summand) for some $b \in V(a)$. The left-right symmetry, as well as the equivalences (1) $\iff$ (5) $\iff$ (7) were also proven independently by D. Khurana, T.Y. Lam, P.P. Nielsen and J. Šter about the same time [113, Theorem 2.13]*, and are now well-known and widely used.

A very different (and probably more visual) proof of the equivalence (1) $\iff$ (6) is given in [161, Theorem 6.1]. It relies on Peirce decomposition and the following trivial fact: a group invertible element $z \in R^{\#}$ is always a unit in eRe for $e = zz^{\#}$, and conversely any element $z \in \mathrm{U}(eRe), e \in E(R)$ is always group invertible (as an element of R).

Theorem 12.2.2 ([161, Theorem 6.1]). Let R be a ring and $a \in R, e \in E(R)$. Then the following statement are equivalent:

(1) There exists $z \in \mathrm{U}(eRe)$ such that $aza = a, zaz = z$;

(2) The Peirce decomposition of a relative to the idempotent e is of the form $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ with $a_1 \in \mathrm{U}(eRe)$ with inverse $z \in \mathrm{U}(eRe)$ and $a_4 = a_3 z a_2$;

(3) $u = a - \bar{e} \in U(R)$ and $au^{-1}a = a$ (a is special clean).

Consequently, we can actually prove that the special clean decompositions are in bijective correspondence with completely regular reflexive inverses.

Corollary 12.2.3 ([161, Corollary 6.2]). Let R be a ring and $a \in R$ be a special clean element. Then there is a bijective correspondence between special clean decompositions and strongly regular reflexive inverses given by $(e, u) \mapsto z = u^{-1}au^{-1}$ with reciprocal $z \mapsto (e = zz^{\#}, u = a - \bar{e})$, where $a = \bar{e} + u = au^{-1}a$ denotes the special clean decomposition.

In particular a is uniquely special clean if and only if it admits a unique reflexive inverse which is also strongly regular.

Another consequence obtained in [161] is that if $eR(1-e) \in J(R)$ for all $e \in E(R)$ (equivalently, idempotents are central modulo the Jacobson radical $J(R)$) then regular elements of R are strongly regular [161, Theorem 6.3]. This result was refined to an equivalence by D. Khurana and P.P. Nielsen in [115, Theorem 3.13]*.

In [141] I study a subclass of the class of special clean elements, which I call *perspective elements*. These perspective elements can also be characterized in terms of group invertible elements as follows: $a \in R$ is (right) perspective iff a is regular and for all $b \in V(a)$, there exists $z \in V(a) \cap R^{\#}$ such that $zR = bR$. Quite surprisingly, it happens that this notion is left-right symmetric as well [141, Theorem 3.4].

Conclusion, open problems and future work

The results of this part on the group inverse prove at least two facts. First, group invertible elements and group inverses are ubiquitous when generalized inverses or idempotents come into play. And among the numerous generalized inverses, they play a very special role. Second, new interesting properties relative to the group inverse are probably still be discovered.

Regarding possible future research inspired by this chapter, I consider the three following ones as promising:

(1) the methodology described in Chapter 9 and used in [159] is not specific to the group inverse, and could probably be applied to the study of the inverse of a product ab along an element in rings (by using the results of [157], see Section 7.2). Or, it could be used to study the inverse along a product;

(2) group-domination has shown a very promising notion to study general (non-unital) rings, and some very special classes of semigroups. It could be interesting to try to perform a generic study of group-domination in the semigroup case;

(3) in Chapter 10, and following the results of [149], we proved that $\mathcal{H}$-commutation (equivalently commutation "modulo $\mathcal{H}$") is a crucial property for the reverse order law of the group inverse. As a is invertible along d may be understood as "a is an inverse of d modulo $\mathcal{H}$" ([158], [148], see Chapter 3 and Chapter 14), it comes into our mind than $\mathcal{H}$-commutativity of certain inverses along an element should be explored;

(4) finally, in Chapter 12 we observe that in the ring case, strongly regular element have a specific additive decomposition; and so do their reflexive inverses, which are exactly the special clean elements. Thus, we can wonder whether one reflexive inverses of special clean elements can be characterized additively (and so on...). For the moment, I have however no idea of what such a characterization would be (the reflexive inverse of a special clean element may not even be clean...)

IV

Algebraic theory of semigroups and structure theorems

Part IV - Algebraic theory of semigroups and structure theorems

This part gathers my contribution to the algebraic study of semigroups (or semigroup (bi)acts). This is mainly obtained by exhibiting some special classes of semigroups, and producing (if possible) structure theorems. More precisely, Chapter 14 defines and studies analogs of known classes of semigroups, but modulo Green's relation $\mathcal{H}$, as suggested by my previous work on the inverse along an element. In Chapter 15, we use extensions of the classical Green's relations to provide structure theorems for some classes of semigroups. Their study as varieties of unary semigroups (in the spirit of universal algebra) is also discussed. In Chapter 16, I explore the ring theoretical notion of perspectivity from a semigroup point of view through the use of chains of associate idempotents. Chapter 17 is of slightly different flavor, since it focuses on the structure of certain monoid biacts (the stable, $\mathcal{J}$-simple monoid biacts). Finally, at the beginning and the end of the part, I focus on other type of results that structure theorems. The part starts with some words on the inverse along an element in semigroups (Chapter 13). And it ends with the study of new partial orders on arbitrary (non regular) semigroups (Chapter 18).

Chapter 13

The inverse along an element in semigroups

Since the beginning of the algebraic study of semigroups, regular semigroups have been recognized as one of the most manageable class of semigroups, due to the abundance of idempotents in such semigroups. And, as soon as they have been introduced, Green's relations have been used to study regularity (hence inner and reflexive inverses). In [146] I followed the same path and used Green's relations to study some inverses but instead of inner or reflexive inverses, I studied the outer inverse in a given $\mathcal{H}$-class. Part II of this memoir is devoted entirely to this notion. In this chapter, I recall the main properties of the inverse along an element in the semigroup case.

13.1) Definition and first properties

The following definitions and results appear in [146] and mary2013generalized. Let S be a semigroup and $a, d \in S$. The *inverse* a^{-d} *of* a *along* d, if it exists, is **the only outer inverse of a in H_d**, the $\mathcal{H}$-class of d. For the inverse of a along d to exist, d must then be regular (for its $\mathcal{H}$-class contains the regular element a^{-d}). If we allow d to vary, then the inverse of a along d may be understood as a parametrization of the outer inverses of a.

Next theorem provides the reader with some necessary and sufficient conditions for a to be invertible along d, as well as different formulas for the inverse a^{-d} of a along d.

Theorem 13.1.1 ([146, Lemma 3 and Theorems 6,7], [158, Theorem 2.2 and Corollary 2.5], [11, Theorem 8.4]*, [103, Theorem 3]*)**.** Let S be a semigroup and $a, d \in S$. Then the following statements are equivalent:
(1) $bab = b$ for some $b \in H_d$ (a is invertible along d);
(2) $bad = d = dab$ for some $b \leq_{\mathcal{H}} d$;
(3) $bab = b$ and $ab = td, ba = dt$ for some $t \in I(d)$ (equiv. $V(d)$).
(4) $ad\,\mathcal{L}\,d$ and H_{ad} is a group;
(5) $da\,\mathcal{R}\,d$ and H_{da} is a group;
(6) $dad\,\mathcal{H}\,d$;
(7) $H_d a H_d = H_d$.
In this case, $a^{-d} = b = d(ad)^{\#} = (da)^{\#}d = d(dad)^{-}d$, for any $dad^{-} \in I(dad)$.

We make some observations:

- if a is in invertible along d, then d is $\mathcal{L}$-related to an idempotent (the identity of H_{ad}). Thus d is regular, so that $I(dad)$ is not empty (equivalently, $d = dab = d\left(a(da)^{\#}\right)d$). The equality $a^{-d} = d(dad)^{-}d$ was proved by Benitez and Boasso [11, Theorem 8.4]*, in the context of rings. But their result carries out straightforwardly to semigroups;
- (3) is [103, Theorem 3]*. It claims that the inverse along an element can be characterized as an outer inverse with prescribed idempotents.
- characterizations (4) and (5) show that group inverses are ubiquitous with regard to generalized inverses. In [77, Theorem 5.17]*, the authors make use of these characterizations to prove that *quotient rings along a function* are *Fountain-Gould quotient rings*;
- The equation $dad\,\mathcal{H}\,d$ characterizes a as a kind of "inner inverse of d modulo $\mathcal{H}$", a statement we took literally and studied carefully in [148]. Equivalently, d may be interpreted as an "outer inverse of a modulo $\mathcal{H}$", a direction followed for instance by Fan et al. [217]*;
- The equality $H_d a H_d = H_d$ claims that $G = H_d$ is a maximal subgroup of the *variant semigroup* $S_a = (S, \cdot_a)$ with multiplication $x \cdot_a y = xay$. Conversely, we can prove that any maximal subgroup G of S_a is of the form H_d, for some d such that a is invertible along d (and the identity of G is a^{-d}).

Interestingly, the case $a^{-d} \in V(a)$ relates to trace products, a direct consequence of Miller and Clifford's theorem 1.1.1.

Theorem 13.1.2 ([146, Corollary 9])**.** Let S be a semigroup and $a, d \in S$. Then the following statements are equivalent:
(1) a is invertible along d and $a^{-d} \in V(a)$ (equiv. $a^{-d} \in I(a)$);
(2) a is invertible along d and d is invertible along a;
(3) ad and da are trace products ($ad \in R_a \cap L_d$ and $da \in R_d \cap L_a$).

From the equivalence (2) $\iff$ (3) in the theorem, we deduce that a^{-a} exists iff $a\,\mathcal{H}\,a^2$, that is a is group invertible. Actually, $a^{-a} = a^{\#}$ in this case [146, Theorem 11] (and many more generalized inverses can be characterized as inverses along a specific element, see [146, Theorem 11] and more generally Section 4.1). It is also well known that the group inverse $a^{\#}$ of a not only commutes but also bicommutes with a. This extends to the inverse along d as follows.

Theorem 13.1.3 ([146, Theorem 10])**.** Let S be a semigroup and $a, d \in S$. Then
(1) $a^{-d} \in \{a, d\}''$ (bicommutant of $\{a, d\}$);
(2) $aa^{-d} \in \{ad\}''$ and $a^{-d}a \in \{da\}''$.

13.2) Inverses along (commuting or bicommuting) idempotents

In this section, we will see that idempotents appear naturally when it comes to commutation properties, a statement that will be made precise below. But inverses along non-commuting idempotents proved also very interesting. The results of this section are also exposed in details in Section 4.3.

Idempotents play a crucial role in semigroup theory. It thus comes to no surprise that inverses along idempotents have interesting properties.

The following lemma (Lemma 4.3.1) regarding inverses along an idempotent is straightforward yet crucial.

Lemma 13.2.1 ([147, Lemma 4])**.** Let $a \in S$ and $e \in E(S)$. Then a is invertible along e iff eae is a unit in the local submonoid eSe, in which case

$$a^{-e} = e(ae)^{\#} = (ea)^{\#}e = (eae)^{\#} = (eae)^{-1}_{[eSe]}.$$

In particular, if a and b are invertible along $e \in E(S)$ then $b^{-e}a^{-e} = (aeb)^{-e}$ ([224, Corollary 2.21]*or [154, Theorem 3.9 (v)]).

To study precisely inverses along idempotents, I found convenient to introduce the following sets, for any $a \in S$ ([147], [151]).

$$\begin{aligned} \Sigma(a) &= \{e \in E(S) | e \leq_{\mathcal{H}} a\}, \\ \Sigma^{\#}(a) &= \{e \in E(S) | eae\mathcal{H}e\}, \\ \Sigma_1(a) &= \{a\}' \cap \Sigma^{\#}(a), \\ \Sigma_2(a) &= \{a\}'' \cap \Sigma^{\#}(a). \end{aligned}$$

By Theorem 13.1.1, $e \in \Sigma^{\#}(a)$ iff a is invertible along e and by [147, Lemma 3] $\Sigma_1(a) = \{a\}' \cap \Sigma(a)$, $\Sigma_2(a) = \{a\}'' \cap \Sigma(a)$.

As any set of idempotents, all these sets are partially ordered by $e \leq f \iff e = ef = fe$. And more specifically, $(\Sigma_2(a), \leq)$ is a semilattice (commutative band) with $e \wedge f = ef$ (product in S) by [147, Proposition 2] (that may thus also denote $(\Sigma_2(a), \wedge)$ or $(\Sigma_2(a), .)$ to emphasize on the min operation rather than on the partial order).

Recall that $W(a)$ is the set of outer (or weak) inverses of a. We let $W_1(a) = \{a\}' \cap W(a)$, $W_2(a) = \{a\}'' \cap W(a)$ and $W^{\#}(a) = S^{\#} \cap W(a)$.

Next theorem proves that there is a bijective correspondence between completely regular (resp. commuting, resp. bicommuting) outer inverses and (resp. commuting, resp. bicommuting) idempotents below a for the $\leq_{\mathcal{H}}$ preorder, and that it extends to an isomorphisms of posets (resp. semilattices) if one consider $W(a)$ as the set of idempotents of the variant semigroup $(S, ._a)$ with product $x._a y = xay$.

Define function $\begin{array}{rccc} \tau: & S^{\#} & \longrightarrow & E(S) \\ & x & \longmapsto & xx^{\#} \end{array}$.

Theorem 13.2.2 ([147, Theorem 3], [151, Lemma 3.1], [151, Corollary 3.1])**.** Function τ restricts to:
(1) an isomorphism $\tau_a^{\#}$ of posets from $(W^{\#}(a), \leq_a)$ onto $(\Sigma^{\#}(a), \leq)$;
(2) an isomorphism τ_a^1 of posets from $(W_1(a), \leq_a)$ onto $(\Sigma_1(a), \leq)$;
(3) an isomorphism τ_a^2 of of semilattices from $(W_2(a), ._a)$ onto $(\Sigma_2(a), .)$.
Their reciprocal associate e to a^{-e}. Also $\tau_a^j(x) = xx^{\#} = ax = xa$ $(j = 1, 2)$.

In summary, τ_a^j is an isomorphism of posets from $(W_j(a), \leq_a)$ onto $(\Sigma_j(a), \leq)$ for $j = \sharp, 1, 2$.

In case there exists a greatest element $M \in \Sigma_j(a)$, then we say that a is $j-$*naturally invertible*, and $b = a^{-M}$ is called the $j-$*natural generalized inverse of* a. Such inverses are introduced and studied in [147], and have been further studied by Kantún-Montiel in [103]*but in the ring and operator algebra context only. In [147], it is notably proved that the natural inverse coincide with the group (or Drazin) inverse if it exists.

Another consequence of this isomorphism is the existence of some "Cline's formula" for commuting outer inverses, relating the commuting outer inverses of a product ab with those of ba.

In the following, we fix $a, b \in S$ and define the function on S $\phi_{b,a} : x \mapsto bx^2a$, and dually $\phi_{a,b}$. It is straightforward that $\phi_{b,a}$ maps $\{ab\}'$ on $\{ba\}'$ and that

$$\phi_{b,a} : (\{ab\}'', ._{ab}) \to (S, ._{ba})$$

is a morphism.

Theorem 13.2.3 ([151, Theorem 2.1 and Corollary 2.2])**.** Function $\phi_{b,a}$ restricts to an isomorphism of posets $j = 1$ (resp. semilattices $j = 2$) from $(W_j(ab), \leq_{ab})$ onto $(W_j(ba), \leq_{ba})$, with reciprocal $\phi_{a,b}$.

As for any $s \in S$, $W_j(s)$ and $\Sigma_j(s)$ are always isomorphic posets (by Theorem 13.2.2), we thus deduce the following Corollary.

Corollary 13.2.4 ([151, Corollary 3.4])**.** Then the following posets $j = 1$ (resp. semilattices $j = 2$) are isomorphic (with their respective structure):

$$W_j(ab) \simeq \Sigma_j(ab) \cap \simeq \Sigma_j(ba) \simeq W_j(ba).$$

Figure 4.1 illustrates Corollary 13.2.4 with commutative diagrams. Each map is an isomorphism of the respective structures ($j = 1, 2$).

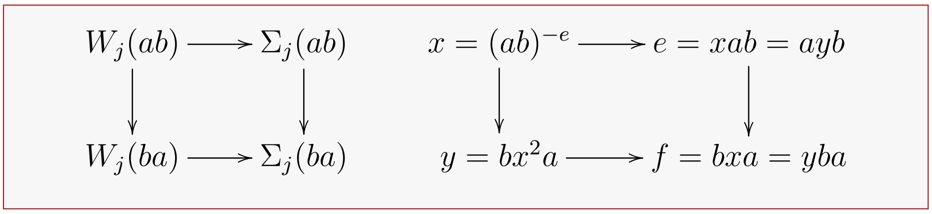


Figure 13.1: Isomorphims of Corollary 13.2.4

The isomorphism goes as follows.

Corollary 13.2.5. Let $e \in \Sigma_j(ab)$, $j = 1, 2$. Then $f = b(ab)^{-e}a \in \Sigma_j(ba)$ with

$$(ba)^{-f} = b\left((ab)^{-e}\right)^2 a$$
$$a(ba)^{-f} = (ab)^{-e}a$$
$$af = ea$$

13.3) Generalized inverses and the Schützenberger category of a semigroup

??

(Most of the results of this section appear also in Chapter 5).

About the same time of the appearance of the inverse along an element in [146], M. P. Drazin defined [51] the (b, c)*-inverse*, that can be seen as an extension of the *Bott-Duffin* (e, f)*-inverse* (which is recovered by letting $b = e$ and $c = f$ be idempotents), and that generalizes the classical generalized inverses (group inverse, Moore-Penrose inverse, Drazin inverse). As a consequence of [154, Corollary 2.5. and Theorem 2.6. (or 2.7.)], the (b, c)-inverse of a in a semigroup S, denoted by $a^{-(b,c)}$ in the sequel, can be characterized as the only outer inverse of a in $R_b \cap L_c$, and we obtain the equality $a^{-d} = a^{-(b,c)}$ for any $b, c, d \in S$ such that $H_d = R_b \cap L_c$.

The elements b, c and d then belong to the same regular $\mathcal{D}$-class $D_{a^{-d}}$, and $b\,\mathcal{R}\,e, c\,\mathcal{L}\,f$ for some idempotents $e, f \in E(S)$, so that finally $a^{-d} = a^{-(b,c)} = a^{-(e,f)}$.

A very interesting feature of the (b, c)-inverse is that it can be understand as a genuine inverse of morphism, in a suitable category. This category is the *Schützenberger category* $\mathbb{D}(S)$ of the semigroup S, as defined by A. Costa and B. Steinberg in [43]*. It has for objects the elements of S, and morphisms are triples $f = (a, x, b)$ with $x \in aS^1 \cap S^1b$. The domain of f is a, its codomain is b and we use the notation $f = a \xrightarrow{x} b$. If $x = au = vb$ and $g = (b, y, c) = b \xrightarrow{y} c$ is a morphism with $y = bw = rc$, then the composition is $g \circ f = a \xrightarrow{x} b \xrightarrow{y} c = a \xrightarrow{vy=xw} c$.

Among all the morphisms from b to c are the *trivial morphims*, of the form $f = c \xrightarrow{x=bac} c$. Next theorem claims that the (b,c)-inverses "are" the inverses of the trivial isomorphisms from b to c (hence the inverses along d are the inverses of the trivial isomorphisms in $Hom(d,d)$).

Theorem 13.3.1 ([154, Theorem 2.7]). Let $a,b,c \in S$. Then a is (b,c)-invertible iff $c \xrightarrow{cab} b$ is an isomorphism of $\mathbb{D}(S)$ ($cab \in R_c \cap L_b$), in which case its inverse morphism is $b \xrightarrow{a^{-(b,c)}} c$.

Not only does this theorem provide a graphical interpretation of the (b,c)-inverse (hence also of the inverse along an element), but it also opens the path to categorical proofs using composition properties. For instance, [154, Corollary 2.8] produces the equality

$$b \xrightarrow{a^{-(b,c)}} c = b \xrightarrow{b} cab \xrightarrow{c} c.$$

Also, we recover that $a^{-e} = (eae)^{-1}_{eSe}$, inverse of eae in the local submonoid eSe, and that $a^{-(e,f)}$ is the unique element $x \in eSf$ such that $x(fae) = e, (fae)x = f$.

This was put to a certain extent in [154] to study reverse order laws (can we compute the inverse of a product by using the product of the inverses?).

We refer to [154] for the statements of the various reverse order laws therein. We only give one result here, to catch a glimpse of the type of results obtained.

Theorem 13.3.2 ([154, Theorem 2.7]). Let $a,w,b,s,t,c \in S$ be such that $a^{-(t,c)}$ and $w^{-(b,s)}$ exist. Then $(aw)^{-(b,c)}$ exists and equals $w^{-(b,s)}a^{-(t,c)}$ iff there exists $e \in E(S)$ such that:
(1) $t \xrightarrow{e} s$ is an invertible morphism;
(2) $caewb = cawb$.
In this case, st is a trace product (and e is the identity of the group $R_t \cap L_s$).

In case the equality $caewb = cawb$ does not hold but st is still a trace product with $e \in R_t \cap L_s$, then the reverse order law becomes $(aew)^{-(b,c)} = w^{-(b,s)}a^{-(t,c)}$ whenever $a^{-(t,c)}$ and $w^{-(b,s)}$ exist.

13.4) Miller and Clifford's theorem revisited

Recall that Miller and Clifford's theorem [166, Theorem 3] (Theorem 1.1.1) states that ab is a trace product ($ab \in R_a \cap L_b$) iff the $\mathcal{H}$-class $H = L_a \cap R_b$ contains an idempotent. We extend this result, and provide applications to the inverse along an element, the (b,c) inverse and the Bott-Duffin inverse.

Theorem 13.4.1 (unpublished)**.** Let S be a semigroup, $a, b \in S$ and $z \in S^1$. Let also $c \in L_a \cap R_b$. Then the following statements are equivalent:
(1) $azb \in R_a \cap L_b$;
(2) $czc\mathcal{H}c$;
(3) $azc\mathcal{H}a$;
(3′) $czb\mathcal{H}b$;
(4) $az\mathcal{R}a$ and $L_{az} \cap R_b$ contains an idempotent;
(4′) $zb\mathcal{L}b$ and $L_a \cap R_{zb}$ contains an idempotent.

Proof. Exchanging the roles of a and b, (1) and (2) are self-dual whereas (3) and (3′) (resp. (4) and (4′)) are dual statements. We prove that (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) and that (2) $\Rightarrow$ (4) $\Rightarrow$ (1). As $c \in L_a \cap R_b$ then $c = xa = by$ and $a = uc, b = cv$ for some $x, y, u, v \in S^1$.

(1) $\Rightarrow$ (2) Assume that $azb \in R_a \cap L_b$. As $azb\mathcal{R}a$ then by left congruence $xazb\mathcal{R}xa$ and $czcv\mathcal{R}c$ so that $czc\mathcal{R}c$. Dually, $czc\mathcal{L}c$.

(2) $\Rightarrow$ (3) Assume that $czc\mathcal{H}c$. Then $c = czct$ for some $t \in S^1$. Thus $a = uc = uczct = azct$ and $a\mathcal{R}azc$. Also as $c\mathcal{L}a$ then by right congruence $czc\mathcal{L}azc$ and finally $a\mathcal{L}c\mathcal{L}czc\mathcal{L}azc$. Thus $a\mathcal{H}azc$.

(3) $\Rightarrow$ (2) Assume that $azc\mathcal{H}a$. Then $a = azct$ for some $t \in S^1$. Thus $c = xa = xazct = czct$ and $c\mathcal{R}czc$. Also as $c\mathcal{L}a$ then by right congruence $czc\mathcal{L}azc$ and finally $c\mathcal{L}a\mathcal{L}azc\mathcal{L}czc$. Thus $c\mathcal{H}czc$.

(2) $\Rightarrow$ (4) Assume that $czc\mathcal{H}c$. Then by Theorem 13.1.1 $(cz)^\#$ exists and $cz\mathcal{R}c\mathcal{R}b$. As $c\mathcal{L}a$ by right congruence $cz\mathcal{L}az$ and $H_{cz} = L_{cz} \cap R_{cz} = L_{az} \cap R_b$ is a group (equivalently contains an idempotent). Finally as $a = uc$ then by left congruence, $az = ucz\mathcal{R}uc = a$.

(4) $\Rightarrow$ (1) Assume that $az\mathcal{R}a$ and $(L_{az} \cap R_b)$ contains an idempotent e. As $az\mathcal{L}e$ and $e\mathcal{R}b$ then $aze = e$ and $eb = b$. It follows that $azb\mathcal{L}eb = b$ by right congruence and $az = aze\mathcal{R}azb$ by left congruence. Finally $a\mathcal{R}azb\mathcal{L}b$.

□

Special cases:

(1) Letting $z = 1$ is the classical theorem;

(2) Letting $a = b = d$, and $z = a$ in Theorem 13.4.1 we recover that $dad\mathcal{H}d$ iff $ad\mathcal{R}d$ and H_{ad} contains an idempotent (Theorem 13.1.1). Moreover, letting $c = a^{-d}$ we recover that if a is invertible along d then $dad\mathcal{H}d$ (since $c\mathcal{H}d$ and $cac = c$);

(3) Letting $z = a$ and $a = c$ we obtain existence criteria for the (b, c)-inverse;

(4) Letting $a = f$ and $b = e$ be idempotents, and $z = a$, we obtain that a is (e, f)-invertible iff $fac = facf$ is a unit in the local monoid fSf for some $c \in L_f \cap R_e$, iff $cae = ecae$ is a unit in the local monoid eSe for some $c \in L_f \cap R_e$;

(5) In particular, a is invertible along e iff $eae \in \mathrm{U}(eSe)$ (this is Lemma 13.2.1);

(6) Let $e = ab, f = ba$ be isomorphic idempotents with (a, b) a regular pair. Then $a \in R_e \cap L_f$. Thus $fze \in R_f \cap L_e$ iff $eza = ezae \in \mathrm{U}(eSe)$ iff $azf = fazf \in \mathrm{U}(fSf)$. In particular, letting $z = 1$, and as $H_a = R_e \cap L_f$ then H_a is a group iff fe is a trace product iff $eae = a^2b \in \mathrm{U}(eSe)$ iff $faf = ba^2 \in \mathrm{U}(fSf)$ (see Corollary 8.1.3, or more generally Section 8.1).

Chapter 14

Classes of semigroups modulo Green's relation $\mathcal{H}$

The study of special classes of semigroups relies in many cases on properties of the set of idempotents, or of regular pairs of elements. Moreover, regarding regular semigroups, the two approaches are usually complementary. For instance, *inverse semigroups* may either be defined as regular semigroups whose idempotents commute, or as semigroups where every element has a unique reflexive inverse. On the other hand, among the regular elements of a semigroup, a special attention as been paid to completely regular elements, and the class of completely regular semigroups is one of the most important class of regular semigroups, as explained in Chapter 2.

Two main ideas have driven my study in [148]:

- First, a completely regular element is known from [73]*to satisfy $a\,\mathcal{H}\,a^2$ (this is Theorem 2.1.1), where $\mathcal{H}$ is Green's relation. Hence, it may be seen as an "idempotent modulo $\mathcal{H}$";
- Second, invertibility along an element can be interpreted as a kind of "regularity modulo $\mathcal{H}$" by [158, Theorem 2.2] (see Chapter 3), since x is invertible along a iff $axa\mathcal{H}a$.

The question of the link between the two notions, completely regular elements and invertibility modulo $\mathcal{H}$, is then natural. It was the purpose of [148] to show that a correspondence exists.

Specifically, [148] introduces *inverses modulo Green's relations* $\mathcal{H}$, and associated classes of semigroups. We follow here the conventions of the memoir rather than those of the article, and use the term "inner inverse" rather than "associate".

Definition 14.0.1 ([148, Definition 1.4])**.** We call a particular solution x to $axa\mathcal{H}a$ an *inner inverse of a modulo* $\mathcal{H}$. If also $xax\mathcal{H}a$, then x is called a *reflexive inverse of a modulo* $\mathcal{H}$, and (a, x) is a *regular pair modulo* $\mathcal{H}$. Finally, we denote the set of all inner inverses of a modulo $\mathcal{H}$ by $I(a)[\mathcal{H}]$, and the set of reflexive inverses of a modulo $\mathcal{H}$ by $V(a)[\mathcal{H}]$.

Some of the results and proofs of the paper are rather technical, but other are more elementary. I present below these latter results, that hopefully will help the reader to fully understand what is meant by "working modulo $\mathcal{H}$".

14.1) Element-wise results

Corollary 14.1.1 ([146, Corollary 9]). Let a, a' be elements of a semigroup S. The following statements are equivalent:
(1) (a, a') is a regular pair modulo $\mathcal{H}$;
(2) $(a')^{-a}$ exists and is a reflexive inverse of a' ($(a')^{-a} \in V(a')$);
(3) aa' and $a'a$ are trace product.

This happens iff $H_a H_{a'} H_a = H_a$ and $H_{a'} H_a H_{a'} = H_{a'}$. While reflexive invertibility modulo $\mathcal{H}$ passes to $\mathcal{H}$-classes, this is not the case for inner invertibility modulo $\mathcal{H}$ in general [148, Example 2.6] (recall that $\mathcal{H}$ is not a congruence in general). For inner invertibility, one has only $aa'a\mathcal{H}a \iff H_a a' H_a = H_a$ [158, Corollary 2.5].

However, the classical properties of inner inverses remain true when working modulo $\mathcal{H}$. Recall that for $a', a'' \in I(a)$, then $a'a, aa' \in E(S)$ and $a'aa'' \in V(a)$. Also, if $a' \in V(a)$ then $a'a = aa' \iff a'\mathcal{H}a$.

Proposition 14.1.2 ([148, Proposition 2.9]). Let a, a', a'' be elements of a semigroup S. Assume that $a', a'' \in I(a)[\mathcal{H}]$. Then $a'a, aa' \in E(S)[\mathcal{H}]$ and $a'aa'' \in V(a)[\mathcal{H}]$. If moreover, $a' \in V(a)[\mathcal{H}]$, then $a'a\mathcal{H}aa' \iff a'\mathcal{H}a$.

In particular regularity and regularity modulo $\mathcal{H}$ represent the same notion [148, Lemma 2.10]. A similar fact happens for inverse and inverse modulo $\mathcal{H}$ semigroups if one uses the "unique reflexive inverse" definition due to the following proposition.

Proposition 14.1.3 ([148, Proposition 2.11]). let S be a semigroup and $a \in S$ be a regular element. The following statements are equivalent:

1. $a', a'' \in V(a) \Rightarrow a' = a''$;
2. $a', a'' \in V(a)[\mathcal{H}] \Rightarrow a'\mathcal{H}a''$.

Two other results of [148] of independent interest are the following ones, that study $\mathcal{H}$-commutation. Recall that by $H(S)$, we denote the union of group $\mathcal{H}$-classes (a.k.a. the set of of completely regular/group invertible elements):

$$H(S) = \bigcup_{e \in E(S)} H_e = S^{\#}.$$

Lemma 14.1.4 ([148, Lemma 3.1]). Let S be a semigroup such that $(\forall a, b \in S)\ a, b \in H(S) \Rightarrow ab\mathcal{H}ba$. Then $E(S)$ is commutative.

Equivalently, the semigroup in Lemma 14.1.4 is such that "idempotents modulo $\mathcal{H}$ commute modulo $\mathcal{H}$."

At the level of (completely regular) elements, $\mathcal{H}$-commutation has the following consequence.

Theorem 14.1.5 ([148, Theorem 4.12 and Lemma 4.14])**.** Let S be a semigroup and $a, b \in H(S)$. The following statements are equivalent.
(1) $ba\mathcal{H}ab$;
(2) $ab, ba \in H(S)$ and $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$.

An more thorough study of the element-wise connections between $\mathcal{H}$-commutation and these reverse order laws has been conducted in [149].

14.2) Completely inverse, $\mathcal{H}$-orthodox, group-closed and E-solid semigroups

Among the classical classes of regular semigroups, the main simple ones are probably *bands* (all elements are idempotents) and *semilattices* (commutative bands), *completely simple* and *completely regular semigroups*, *inverse semigroups* (each element has a unique reflexive inverse, equiv. the semigroup is regular and idempotents commute), *Clifford semigroups* (completely regular and inverse, equiv. the semigroup is regular and idempotents are central) and *orthodox semigroups* (the semigroup is regular and idempotents form a subsemigroup). All of them admit characterizations in terms of idempotents or inner or reflexive inverses, and sometimes distinct but equivalent ones. The main contribution of [148] is to define their analogs modulo $\mathcal{H}$ **for all equivalent characterizations**, check whether they remain equivalent or not, and more generally study more closely their connections with existing classes of semigroups. In this memoir, we complete a little bit [148] by adding results for bands, semilattices and completely regular semigroups modulo $\mathcal{H}$. A subset A of a semigroup S is *$\mathcal{H}$-commutative* if $(\forall a, b \in A), ab\mathcal{H}ba$. Finally, a semigroup is *E-solid* if if $(\forall e, f, g \in E(S))\ f\,\mathcal{R}\,e\,\mathcal{L}\,g \Rightarrow (\exists h \in E(S))\ f\,\mathcal{L}\,h\,\mathcal{R}\,g$ [80]*, iff the subsemigroup generated by its idempotents is completely regular (a union of groups) [80, Theorem 3]*.

Next table subsumes the definitions and results of [148] (explanations are given below).

Characterization	**Name**	**Char. modulo $\mathcal{H}$**	**Name**
$(\forall a \in S)\ a \in E(S)$	band	$(\forall a \in S)\ a \in H(S)$	completely regular
$(\forall a \in S)\ a \in E(S)$, $E(S)$ commutative	semilattice	$(\forall a \in S)\ a \in H(S)$, $H(S)$ $\mathcal{H}$-commutative	Clifford
$(\forall a \in S)\ V(a) \cap \{a\}' \neq \emptyset$	completely regular	$(\forall a \in S)\ V(a)[\mathcal{H}] \cap \{a\}'[\mathcal{H}] \neq \emptyset$	completely regular
$E(S)E(S) \subseteq E(S)$	orthodox	$H(S)H(S) \subseteq H(S)$	$\mathcal{H}$-orthodox
$(\forall a \in S)\ V(a)$ is a singleton	inverse	$(\forall a \in S)\ V(a)[\mathcal{H}]$ is a single $\mathcal{H}$-class	inverse
$E(S)$ is commutative	inverse	$H(S)$ is $\mathcal{H}$-commutative	completely inverse
$(\forall a, b \in S)\ I(b)I(a) \subseteq I(ab)$	orthodox	$(\forall a, b \in S)\ I(b)[\mathcal{H}]I(a)[\mathcal{H}] \subseteq I(ab)[\mathcal{H}]$	$\mathcal{H}$-orthodox
$(\forall a, b \in S)\ V(b)V(a) \subseteq V(ab)$	orthodox	$(\forall a, b \in S)\ V(b)[\mathcal{H}]V(a)[\mathcal{H}] \subseteq V(ab)[\mathcal{H}]$	$\mathcal{H}$-orthodox
$(\forall e \in E(S))\ V(e) \subseteq E$	orthodox	$(\forall h \in H(S))\ V(h)[\mathcal{H}] \subseteq H(S)$	E-solid

- Bands modulo $\mathcal{H}$ correspond to completely regular semigroups by Green's theorem 2.1.1. Semilattices modulo $\mathcal{H}$ correspond to Clifford semigroups by [148, Theorem 4.15];

- That completely regular and completely regular modulo $\mathcal{H}$ semigroups coincide follows from Corollary 14.1.1;
- Completely inverse semigroups are inverse by [148, Lemma 3.1]; On the contrary, $\mathcal{H}$-orthodox semigroups need not be orthodox [148, Example 4.9];
- The various equivalence for $\mathcal{H}$-orthodox semigroups are the content of [148, Theorem 4.3 and Lemma 4.14];
- The last equivalence ($(\forall h \in H(S))\ V(h)[\mathcal{H}] \subseteq H(S)$ iff S is E-solid) is [148, Theorem 4.6].

14.3) Centrality, crypticity

Pushing further the study, I was able to provide more characterizations of completely inverse semigroups. One uses the centralizer $Z(E(S))$ [148, Corollary 5.3], and another crypticity [148, Theorem 5.12]. Recall that a semigroup is *cryptic* if Green's relation $\mathcal{H}$ is a congruence; Clifford semigroups are cryptic with central idempotents ($Z(E(S)) = S$). All together, I proved that the following statements are equivalent:

(1) S is completely inverse (S is regular and $H(S)$ is a $\mathcal{H}$-commutative set);
(2) S is regular and $H(S)$ is a Clifford semigroup;
(3) S is regular and $H(S) = Z(E(S))$;
(4) S is $\mathcal{H}$-orthodox and $\mathcal{H}_{H(S)}$ is a commutative congruence;
(5) S is inverse and $\mathcal{H}$-orthodox;
(6) S is inverse and $\mathcal{H}$ is a congruence (S is cryptic inverse);
(7) $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is inverse.

In addition, is it not difficult to deduce from these results that *$\mathcal{H}$-Clifford semigroups*, defined as semigroups regular modulo $\mathcal{H}$ and such that idempotents modulo $\mathcal{H}$ are central modulo $\mathcal{H}$ (that is, as regular semigroups S such that $(\forall h \in H(S), \forall a \in S)\ ah\mathcal{H}ha$), are just Clifford semigroups (which are also the regular $\mathcal{H}$-commutative semigroups by [2, Theorem 5.1]*or [149, Theorem 2.7]).

Corollary 14.3.1 (unpublished)**.** Let S be a semigroup. Then the following statements are equivalent:
(1) S is regular and completely regular elements are central modulo $\mathcal{H}$;
(2) $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is a semilattice;
(3) S is completely regular and completely inverse;
(4) S is regular and $\mathcal{H}$-commutative;
(5) S is a Clifford semigroup.

To my very surprise, I just discovered very recently an article due to M. Petrich and also published in 2014 (as [148]) in which some previous equivalences involving cryptic inverse semigroups were also proved [192, Theorem 4.1 and Corollary 4.3]*. The motivation and approach of [192]*is however very different in nature.

14.4) Quasivarieties

Regarding universal algebra, it is shown in [148] that the class of completely inverse semigroups is not a *variety* of (2, 1)-algebras (algebras with the two operations of multiplication and inversion). Indeed, if we take X an inverse not completely inverse semigroup, and let $S = (\mathcal{F}X, f)$ be the free inverse semigroup on X, then $\mathcal{F}X$ is combinatorial hence completely inverse, whereas its homomorphic image X is not. The class of completely inverse semigroups is however a *pseudoelementary class* of type (2, 1) closed under subalgebras (inverse subsemigroups) and direct products and, as such, a *quasivariety* of type (2, 1). Indeed, it may be simply defined by adding to the identities defining the variety of inverse semigroups the quasi-identity

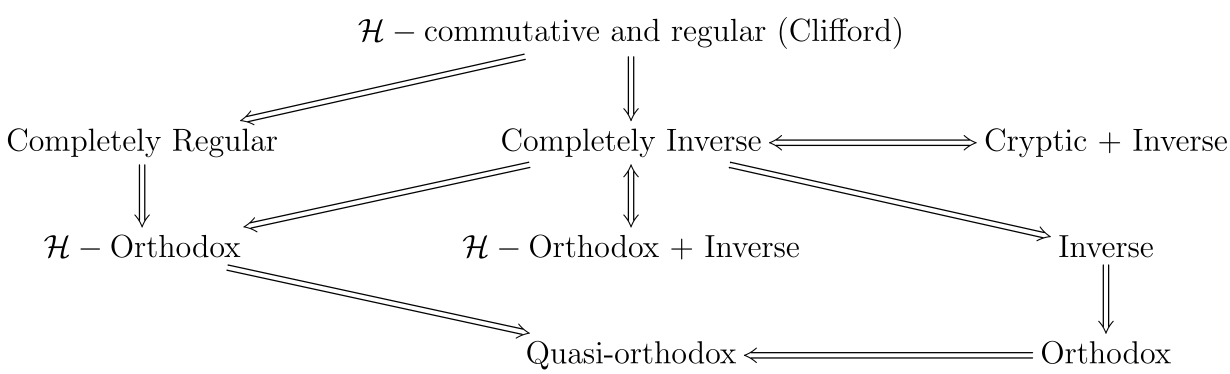
$$xx^{-1} = x^{-1}x \Rightarrow xyy^{-1} = yy^{-1}x$$

(Completely regular elements commute with idempotents).

14.5) Regular semigroups

We have the following implications between the different types of regular semigroups studied in [148] (a semigroup is quasi-orthodox [220]*iff it is regular and E-solid [81]*).

Figure 1: Regular semigroups.



14.6) Non-regular semigroups

The same kind of implications exist for non-regular semigroups, where:

(1) S is an *E-semigroup* if $E(S)$ is a subsemigroup;

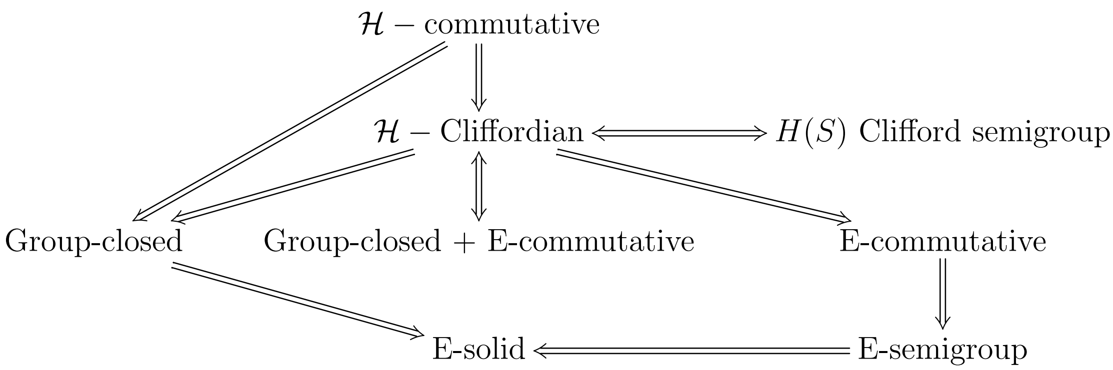
(2) S is *E-commutative* if $E(S)$ is commutative;

(3) S is *group-closed* [5]*if $H(S)$ is a subsemigroup;

(4) S is *$\mathcal{H}$-commutative* if $(\forall a, b \in S)\ ab\mathcal{H}ba$;

(5) S is *$\mathcal{H}$-Cliffordian* if $H(S)$ is $\mathcal{H}$-commutative.

Figure 2: Non-regular semigroups.



Chapter 15

Completely (E, H_E)-abundant semigroups

In this chapter, we continue our study of certain special classes of semigroups. However, contrary to Chapter 14 (and [148]), where the main characters were generalized inverses or/and idempotents, our main tool here will be Green's (extended) relations. Precisely, I will present the results of [150], that studies analogs to completely regular, completely simple and Clifford semigroups, but defined in terms of a distinguished subset of idempotents E (whose elements act as minimal left and right identities) and its associated Green's extended relations $\widetilde{\mathcal{K}}_E$. Also, it will give the opportunity to introduce a new point of view, that of universal algebra. Indeed, we will see that instead of working with plain semigroups and consider certain -Green's relations based- properties, it may be convenient to consider them as unary semigroups (algebras of type $(2,1)$). We will then be able to prove that all classes studied are varieties (of unary semigroups) and consider their defining equations.

15.1) Green's extended relations $\widetilde{\mathcal{K}}_E$

Let S be a semigroup. For any equivalence relation σ on S, $A \subseteq S$ is $\sigma-$saturated (or saturated by σ) if A is a union of σ-classes, or equivalently, $(a,b) \in \sigma$ and $a \in A \Rightarrow b \in A$. The semigroup S is $\sigma-$abundant ((E,σ)-abundant) if every $\sigma-$class contains idempotents of S (contains elements of E).

We will make use of the Green's extended preorders and relations in a semigroup. If $\leq_{\mathcal{K}}$ is a preorder, then $a \,\mathcal{R}\, b \iff \{a \leq_{\mathcal{K}} b \text{ and } b \leq_{\mathcal{K}} a\}$, and $K(a) = \{b \in S,\ b\mathcal{K}a\}$ denotes the $\mathcal{K}$-class of a (this notation is preferred to the most classical K_a in this chapter to avoid multiple subscripts).

For elements a and b of S, $\leq_{\widetilde{\mathcal{L}}_E}$ and $\leq_{\widetilde{\mathcal{R}}_E}$ are defined (see for instance [72]*, [132]*) by

$$a \leq_{\widetilde{\mathcal{L}}_E} b \iff \{(\forall e \in E)\ be = b \Rightarrow ae = a\};$$
$$a \leq_{\widetilde{\mathcal{R}}_E} b \iff \{(\forall e \in E)\ eb = b \Rightarrow ea = a\}.$$

(For $E = E(S)$ we forget the subscript.)

As is well known, $\mathcal{L} \subseteq \widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{L}}_E$ [96, lemma 4.1]*and $\mathcal{L} = \widetilde{\mathcal{L}}(= \widetilde{\mathcal{L}}_{E(S)})$ on regular semigroups [96, lemma 4.14]*. Contrary to $\mathcal{L}$ (resp. $\mathcal{R}$), $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{R}}$) is not a right (resp. left) congruence in general. In particular, $\widetilde{\mathcal{L}}_E$ ($\widetilde{\mathcal{R}}_E$) needs not to be a right (left) congruence. If this is the case, we will say that S is $\widetilde{\mathcal{L}}_E-$ (resp. $\widetilde{\mathcal{R}}_E-$) congruent, or, following Fountain et al. [66]*, that S satisfies (CL) (resp. (CR)). A semigroup which satisfies conditions (CL) and (CR) is also said to satisfy the congruence conditions [133]*.

The intersection of preorders (resp. equivalence relations) $\leq_{\widetilde{\mathcal{L}}_E}$ and $\leq_{\widetilde{\mathcal{R}}_E}$ is also a preorder (resp. equivalence relation), denoted by $\leq_{\widetilde{\mathcal{H}}_E}$ (resp. $\widetilde{\mathcal{H}}_E$). Recall that relations $\mathcal{L}$ and $\mathcal{R}$ commute, so that their join $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$ is just $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. The extended relations $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}_E$ are defined analogously as the join of the extended relations, but not as their product since $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ do not commute in general. Finally, there is a last relation $\widetilde{\mathcal{J}}_E$ (see [221]*) based on saturated ideals in replacement of the classical relation $\mathcal{J}$ (equality of principal ideals).

15.2) $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups

Let S be a semigroup, and $E \subseteq E(S)$ a distinguished subset of idempotents. We say that S is $(E, \widetilde{\mathcal{H}}_E)$-*abundant* if any element of S is $\widetilde{\mathcal{H}}_E$-related to an element of E. These semigroups (with $E = U$) were formerly named weakly U-superabundant semigroups. S is *completely* $(E, \widetilde{\mathcal{H}}_E)$-abundant if moreover $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ are right and left congruences respectively.

15.2.1) Completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups

We first present some existing results regarding completely $(E, \widetilde{\mathcal{H}}_E)$-*abundant* semigroups. These results show that, to some extent, completely $(E, \widetilde{\mathcal{H}}_E)$-*abundant* semigroups behave like their classical counterpart (completely regular semigroups). By [200, Lemma 2.1 and Theorem 2.2]*, on such semigroups $\widetilde{\mathcal{J}}_E = \widetilde{\mathcal{D}}_E$ and is a semilattice congruence, so that there is a semilattice decomposition with each component J_α completely $(E_\alpha, \widetilde{\mathcal{H}}_{E_\alpha})$-abundant and $\widetilde{\mathcal{J}}_{E_\alpha}$-simple. Moreover, these components are Rees matrix semigroups [137]*. And finally, one can construct a completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup from a given semilattice Y and a family $J_\alpha, \alpha \in Y$ of completely $(E, \widetilde{\mathcal{H}}_E)$-abundant, $\widetilde{\mathcal{J}}_{E_\alpha}$-simple semigroups [221]*.

Regarding these results, it happens that the hypothesis involved in both the semilattice decomposition theorem and the semilattice composition theorem are rather strong and difficult to handle. First, they rely on the intricate condition $\widetilde{\mathcal{J}}_E$. And second, the congruence condition has to be checked. In [150], I was able to improve the previous results by using bisimplicity ($\widetilde{\mathcal{D}}_E$-simplicity) instead of simplicity ($\widetilde{\mathcal{J}}_E$-simplicity), and a simple property (Π):

$$(\Pi)(\forall a \in S, \forall e, f, g \in E)\ eaf \in M_g \Rightarrow egf = g.$$

where M_g will be a specific monoid that depends on the context.

Theorem 15.2.1 ([150, Theorem 2.6])**.** Let S be a $(E,\widetilde{\mathcal{H}}_E)$-abundant semigroup. Then the following statements are equivalent:
(1) The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are right and left congruences;
(2) The relation $\widetilde{\mathcal{D}}_E$ is a semilattice congruence;
(3) The relation $\widetilde{\mathcal{D}}_E$ is a congruence.

The main consequence of this theorem is that a $(E,\widetilde{\mathcal{H}}_E)$-abundant, $\widetilde{\mathcal{D}}_E$-simple semigroup is automatically completely $(E,\widetilde{\mathcal{H}}_E)$-abundant. Consequently, we chose in [150] to name them simply *completely E-simple semigroups.* As $\widetilde{\mathcal{J}}_E = \widetilde{\mathcal{D}}_E$ on such semigroups, they are precisely the completely $(E,\widetilde{\mathcal{H}}_E)$-abundant, $\widetilde{\mathcal{J}}_E$-simple semigroups. We will see in next section that given a semigroup S, it can be completely E-simple for at most one choice of idempotents E [150, Corollary 3.10].

We now give a new version of the semilattice decomposition.

Theorem 15.2.2 ([150, Theorem 2.6])**.** A semigroup S is completely $(E,\widetilde{\mathcal{H}}_E)$-abundant if and only if it is a semilattice Y of $(E_\alpha,\widetilde{\mathcal{H}}_{E_\alpha})$-abundant, $\widetilde{\mathcal{D}}_{E_\alpha}$-simple semigroups with Π:

$$(\Pi)(\forall a_\gamma \in S_\gamma, \forall e_\alpha, f_\beta, g_{\alpha\gamma\beta} \in E)\ e_\alpha a_\gamma f_\beta \in \widetilde{H}_{E_{\alpha\gamma\beta}}(g_{\alpha\gamma\beta}) \Rightarrow e_\alpha g_{\alpha\gamma\beta} f_\beta = g_{\alpha\gamma\beta}.$$

Then, in [150] we use this theorem to prove a second semilattice composition theorem [150, Theorem 2.8], very close to the one of Petrich for completely regular semigroups [190, Theorem 2.3]*, and whith an additional condition easier to handle than in [221]*. Due to its technical nature, we refer directly to [150] for a precise statement of Theorem 2.8 therein. As a corollary to [150, Theorem 2.8] we obtain the following result.

Corollary 15.2.3 ([150, Corollary 2.9])**.** Let S be a completely $(E,\widetilde{\mathcal{H}}_E)$-abundant semigroup. Then $\widetilde{H}(E) = \dot{\bigcup}_{e\in E}\widetilde{H}_E(e)$ is a completely regular subsemigroup of S.

15.2.2) Completely E-simple semigroups

We say that $e \in E$ is *primitive* (within E) if $(\forall f \in E)\ ef = f = fe \Rightarrow e = f$ (e is primitive if it is a minimal element of E with respect to the natural partial order). Primitive idempotents play a special in the theory of completely simple semigroups: a semigroup is completely simple iff it is completely regular with some (all) idempotents primitive.

A similar result holds for completely E-simple semigroups.

Theorem 15.2.4 ([150, Theorem 3.8]). Let S be a $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup. Then it is $(E, \widetilde{\mathcal{D}}_E)$-simple iff some (all) idempotent(s) of E is (are) primitive.

In [150, Theorem 3.8], only the "all" case is stated and proved. But the proof works without change in the "some" case (all idempotents are $\widetilde{\mathcal{D}}_E$ related to that fixed primitive $e \in E$ in that case).

As a consequence, we obtain that **being completely E-simple is an intrinsic property of** S. Indeed, the only possible set E is then $E = Max$, set of maximal idempotents of $E(S)$ (with respect to the natural partial order), as proved in [150, Corollary 3.10].

Another characterization of completely simple semigroups is as regular semigroups that are disjoint union of their local submonoids. This extends to completely E-simple semigroups.

Corollary 15.2.5 ([150, Corollary 3.12]). Let S be a semigroup, and E a distinguished set of idempotents such that $S = \dot{\bigcup}_{e \in E} eSe$. Assume moreover that E is such that

$$(\Pi'')(\forall e, f \in E)\ fef = fe \Rightarrow fe \in E \text{ and } fef = ef \Rightarrow ef \in E.$$

Then S is completely E-simple. Conversely, every completely E-simple semigroup is the disjoint union of its local submonoids $eSe, e \in E$, and satisfies

$$(\Pi')(\forall e, f \in E)\ fef = fe \Rightarrow fe = f \text{ and } fef = ef \Rightarrow ef = f.$$

In a completely E-simple semigroup, each local submonoid $eSe, e \in E$ coincide with the $\widetilde{\mathcal{H}}_E$-class $\widetilde{H}_E(e)$. Also, products are trace products (with respect to the extended Green's relations: $(\forall a, b \in S)\ ab \in \widetilde{R}_E(a) \cap \widetilde{L}_E(b)$) and $\widetilde{\mathcal{H}}_E$ is a congruence [150, Corollary 3.14].

And finally, there is a Rees matrix representation theorem [150, Corollary 3.16]. For regular semigroups we recover (quite unexpectedly) some results of Hickey on regularity-preserving elements [93, Theorems 5.2 and 5.5]*.

15.2.3) Union of monoids

The general case of $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups such that each $(E, \widetilde{\mathcal{H}}_E)$-class is a monoid is also of interest. By definition such semigroups are union of monoids, but the converse needs not be true [150, Example 3.3] (contrary to the classical case, where completely regular semigroups are exactly unions of groups). Once again, a type Π condition appears in order to get a converse statement.

Theorem 15.2.6 ([150, Theorem 3.2]). Let $S = \bigcup_{e\in E} M_e$ be a disjoint union of monoids M_e with identity e such that

$$(\Pi)(\forall a \in S, \forall e, f, g \in E)\ eaf \in M_g \Rightarrow egf = g.$$

Then S is $(E, \widetilde{\mathcal{H}}_E)$-abundant. Conversely, any $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup such that each $(E, \widetilde{\mathcal{H}}_E)$-class is a monoid is a union of monoids with (Π).

In case of union of groups, the extra (Π) condition is always fulfilled: if $eaf \in H(g)$ then $g = eaf(eaf)^{\#} = egf$.

15.2.4) E-Clifford restriction semigroups

Finally, recall that a Clifford semigroup can be characterized as either a completely regular semigroup whose idempotents commute (equiv. form a semilattice), or as an inverse semigroup with central idempotents, and that on such semigroups relation $\mathcal{H}$ is a congruence. Define an E-Clifford restriction semigroup as a semigroup S with distinguished subset of idempotents $E \subseteq E(S)$ such that: (1) Elements of E are central idempotents; (2) Every $\widetilde{\mathcal{H}}_E$-class $\widetilde{H}_E(a)$ contains a (necessarily unique) idempotent; (3) The relation $\widetilde{\mathcal{H}}_E$ is a congruence.

Corollary 15.2.7 ([150, Corollary 5.5 and Theorem 5.8]). Let S be a semigroup, and $E \subseteq E(S)$ be a set of idempotents. Then the following statements are equivalent:
(1) S is an E-Clifford restriction semigroup;
(2) S is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant and E is a semilattice;
(3) S is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant and idempotents of E commute;
(4) S is a strong semilattice E of monoids $M_e, e \in E$, with identities e;
(5) S is a semilattice E of monoids $M_e, e \in E$, with identities e.

15.3) Varieties of unary semigroups

Semigroups are a particular type of (2)-algebras (sets endowed with a binary operation), those whose operation is associative. In this section, we will consider $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups as unary semigroups, that is $(2, 1)$-algebras endowed with an associative binary operation and a unary operation. Given a $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup S, the unary operation associates to any $x \in S$ the unique idempotent $x^+ = e \in E$ such that $x\widetilde{\mathcal{H}}_E e$. For any unary semigroup S, we also pose $S^+ = \{x^+|x \in S\}$ and $\sigma^+ = \{(x, y) \in S \times S | x^+ = y^+\}$. For any set of identities $\{1, \ldots, n\}$, $\mathcal{V}(i_1, \ldots, i_k)$ denotes the variety of unary semigroups that satisfy the identities $(i_1, ...i_k)$.

For instance, the variety of *left restriction semigroups* has attracted lots of interest (see [96]*for a very interesting survey on the topic). A unary semigroup $(S, ., ^+)$ is a left restriction semigroup if its unary operation satisfies the identities

$$x^+x = x, x^+y^+ = y^+x^+, (x^+y)^+ = x^+y^+, xy^+ = (xy)^+x.$$

Right restriction semigroups are defined dually, and a bi-unary semigroup $(S, ., ^+, ^*)$ with $(S, ., ^+)$ (resp. $(S, ., ^*)$ a left (resp. right) restriction semigroup is a *restriction semigroup*. It is proved in [150] that E-Clifford restriction semigroups are indeed (left) restriction semigroups.

Proposition 15.3.1 ([150, Propositions 5.1 and 5.2]). Let S be an E-Clifford restriction semigroup. For any $x \in S$, let $x^+ = e$ be the unique idempotent $e \in E$ such that $x\widetilde{\mathcal{H}}_E e$. Then $(S, ., ^+)$ is a left and right restriction semigroup such that its set of projection $S^+ = E$ is a semilattice of central idempotents (in S); In particular, $(S, ., ^+, ^+)$ is a restriction semigroup.
Conversely, any left (equiv. right) restriction semigroup such that its set of projection $S^+ = E$ is a semilattice of central idempotents (in S) is an E-Clifford restriction semigroup with $E = S^+$.

Moreover, [150, Theorem 5.3] proves that $(S, ., ^+, ^+)$ is a restriction semigroup iff $(S, ., ^+)$ is a left restriction semigroup such that its set of projection $S^+ = E$ is a semilattice of central idempotents (in S).

We consider the following identities on a unary semigroup $(S, ., ^+)$:

$$x^+x = x \tag{15.1}$$
$$xx^+ = x \tag{15.2}$$
$$(xy^+)^+y^+ = (xy^+)^+ \tag{15.3}$$
$$y^+(y^+x)^+ = (y^+x)^+ \tag{15.4}$$
$$x^+x^+ = x^+ \tag{15.5}$$
$$x^+xx^+ = x \tag{15.6}$$
$$x^+(x^+zy^+)^+y^+ = (x^+zy^+)^+ \tag{15.7}$$
$$x^{++} = x^+ \tag{15.8}$$
$$x^+(xy)^+y^+ = (xy)^+ \tag{15.9}$$
$$(x^+y)(xy)^+ = x^+y \tag{15.10}$$
$$(yx)^+(yx^+) = yx^+ \tag{15.11}$$
$$(xy)^+ = (x^+y)^+ \tag{15.12}$$
$$(yx)^+ = (yx^+)^+ \tag{15.13}$$
$$(xy)^+ = (x^+y^+)^+ \tag{15.14}$$
$$x^+(yx)^+ = x^+ \tag{15.15}$$
$$(xy)^+x^+ = x^+ \tag{15.16}$$
$$y^+(yx)^+ = (yx)^+ \tag{15.17}$$
$$(xy)^+y^+ = (xy)^+ \tag{15.18}$$
$$x^+y = yx^+ \tag{15.19}$$
$$(xy)^{++} = x^+y^+ \tag{15.20}$$

The main result of [150] regarding universal algebra is that not only E-Clifford restriction semigroups but all classes previously studied form varieties of unary semigroups. In particular, as any variety of $(2,1)$-algebras, these classes are stable under direct product, homomorphic images and subalgebras by Birkoff theorem.

Theorem 15.3.2 ([150, Theorem 3.2, Propositions 5.1])**.**

(1) $S^+\mathcal{A} = \mathcal{V}(1,2,3,4) = \mathcal{V}(5,6,7) = \mathcal{V}(1,2,8,9)$ is the variety of unary $(S^+, \widetilde{\mathcal{H}}_{S^+})$-abundant semigroups;
(2) $\mathcal{C}S^+\mathcal{A} = \mathcal{V}(1,2,3,4,10,11) = \mathcal{V}(1,2,9,12,13)$ is the subvariety of unary completely $(S^+, \widetilde{\mathcal{H}}_{S^+})$-abundant semigroups;
(3) $S^+\mathcal{CG} = \mathcal{V}(1,2,3,4,14)$ is the subvariety of unary completely $(S^+, \widetilde{\mathcal{H}}_{S^+})$-abundant, $\widetilde{\mathcal{H}}_{S^+}$-congruent semigroups ($S^+$-cryptogroups);
(4) $\mathcal{C}S^+\mathcal{S} = \mathcal{V}(1,2,15,16,17,18) = \mathcal{V}(1,2,9,15,16)$ is the subvariety of unary completely S^+-simple semigroups;
(5) $S^+\mathcal{C}l\mathcal{R} = \mathcal{V}(1,19,20)$ is the subvariety of S^+-Clifford restriction semigroups.

Moreover, $\mathcal{C}S^+\mathcal{S} \subset S^+\mathcal{CG} \subset \mathcal{C}S + \mathcal{A} \subset S^+\mathcal{A}$, and $S^+\mathcal{C}l\mathcal{R} \subset S^+\mathcal{CG}$.

Consequently, we deduced [150, Corollary 5.6] that a unary semigroup is a Clifford restriction semigroup iff it is a subdirect product of restriction monoids and restriction monoids with a zero added. Observe that such a result fails for plain E-Clifford restriction semigroups. Indeed [150, Example 5.7] provides an example of a subdirect product of monoids and monoids with a zero added that is not completely $(E, \widetilde{\mathcal{H}}_E)$-abundant (let alone an E-Clifford restriction semigroup).

Finally, by [150, Proposition 4.11] these semigroups can be partially ordered by

$$(\forall a, b \in S)\ a\sigma b \iff a = a^+b = ba^+.$$

Chapter 16

Chains of associate idempotents and chained semigroups

In [117], [139], [141] and [156] (see also [112]*, [115]*, [116]*), we propose a new path to study perspectivity of modules, by studying **chains of associate idempotents** in the endomorphism ring of the module. It happens that such chains can be studied without any reference to module and rings properties, in a pure semigroup setting. In this chapter, I propose to study semigroup analogs to certain interesting classes of rings; while this pure semigroup road seems promising, it is however just at its beginning and may be very challenging in full generality.

16.1) Associate idempotents and n-chained semigroups

Let S be a semigroup and $e, f \in E(S)$. Then e, f are *left (resp right) associates* if $ef = e$ and $fe = f$ (resp. $ef = f, fe = e$) and we write $e \sim_\ell f$ (resp. $e \sim_r f$). This notion appears in the early fifties [40]*, and the preorders induced by left/right association of idempotents ($e\omega_\ell f$ if $ef = e$ and $e\omega_r f$ if $fe = e$) are notably a primitive notion regarding *biordered sets* ([60]*, [177]*, [178]*, [188]*). Left (resp. right) association $\sim_\ell$ (resp. $\sim_r$) was for instance denoted by $\overset{l}{\approx}$ (resp. $\overset{r}{\approx}$) in [177]*and by $\succ\!\!-\!\!\prec$ (resp. $\leftrightarrow$) in [60]*. It was rediscovered by Nielsen et. al. in the context of rings in [184]*.

Association of idempotents is closely linked with Green's relations. For any two idempotents $e, f \in E(S)$, it holds that $e\,\mathcal{L}\,f \iff e \sim_\ell f$, $e\,\mathcal{R}\,f \iff e \sim_r f$, $e\,\mathcal{H}\,f \iff e = f$ and $e\mathcal{D}f \iff eS \simeq fS$ (as right S-acts, and we say that e, f are *isomorphic*, denoted by $e \simeq f$). Alternatively, $e \simeq f$ iff $e = ab$ and $f = ba$ for some $a, b \in S$ (and we can always choose a, b to be reflexive inverses).

Let $n \in \mathbb{N}$; following [115]*and [116]*we define a *left n-chain* from e to f as a sequence of $n+1$ idempotents $e_0; e_1; \dots ; e_n \in E(S)$ such that

$$e = e_0 \sim_\ell e_1 \sim_r e_2 \sim_\ell \cdots e_n = f.$$

The number n is *the length of the chain*. Right n-chains are defined dually. When n is small, such as $n = 2$ or $n = 3$, we will write $e \sim_{\ell r} f$, respectively $e \sim_{\ell r \ell} f$, and more generally, we will write $e \sim_{(\ell r)^p} f$ (resp. $e \sim_{(\ell r)^p \ell} f$ or $e \sim_{\ell (r\ell)^p} f$) to denote that e and f are connected by a left chain of length $2p$ (resp. $2p+1$). We define *right n-chains* dually and write $e \approx f$ to denote that e and f are connected by some (left or right) association chain. Relation $\approx$ is nothing but the transitive closure of the union of $\sim_\ell$ and $\sim_r$ (and as such an equivalence relation).

Following [117], [139] and [156], and still using the terminology of [116]*, we also define the following properties:

(1) S is *left n-chained* if any two isomorphic idempotents are connected by a left n-chain. In this case we also say that S satisfies $\mathcal{P}_\ell(n)$. Property $\mathcal{P}_r(n)$ is defined dually;

(2) S is (strongly) *n-chained* if any two isomorphic idempotents are connected by both a left and a right n-chain;

(3) S is *weakly n-chained* if any two isomorphic idempotents are connected by *either* a left and a right association chain of length n;

(4) S is π-chained (or $\approx$-chained) if any two isomorphic idempotents are connected by some (left or right) association chain.

By definition, n-chaining is a local property that depends on the regular $\mathcal{D}$-classes only.

We can use association chains to refine the notion of regularity as follows [139]. We say that $a \in S$ is *n-chained regular* if it is regular and for all $b \in V(a)$, ab and ba are right n-chained. It is *n-anti-chained regular* if it is regular and for all $b \in V(a)$, ab and ba are left n-chained. The terminology comes from the following fact: for any $b \in V(a)$, $abS = aS$ so that starting **forward** in the chain, it makes sense to consider first equality of **right** principal ideals, whereas $Sba = Sa$ so that starting **backward** in the chain, it makes sense to consider first equality of **left** principal ideals. By [139, Proposition 2.2], a semigroup is right (resp. left) n-chained iff regular elements are n-chained regular (resp. n-anti-chained regular).

If $n = 2p$ is even, then any two idempotents $e, f \in E(\mathcal{S})$ satisfy $e \sim_{r\ell}^p f$ iff $f \sim_{\ell r}^p e$ so that left and right $2p$-chained semigroups coincide. This is not the case for left/right $2p+1$-chained semigroups in general.

Example 16.1.1 ([139, Example 2.4])**.** Let S be a left zero semigroup ($\forall a, b \in S,\ ab = a$) with at least two distinct elements e, f. Then any two elements are idempotents and left associates and S is left 1-chained. But e, f are isomorphic ($ef = e, fe = f$) and not right associates (otherwise they would be equal), and S is not right 1-chained. Also, e is 1-anti-chained regular but not 1-chained regular.

A cornerstone of the next results is the relationship between chains of different length between product of reflexive inverses.

Theorem 16.1.1 ([139, Theorem 2.5]). Let S be a semigroup, $a \in \text{reg}(S)$ and $p \in \mathbb{N}$. Then the following statement are equivalent:
(1) $ab \sim_{r\ell}^{p} \circ \sim_r ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);
(2) $ab \sim_{\ell r}^{p} ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);
(3) $ab \sim_{r\ell}^{p+1} ba$ for all $b \in V(a)$ (equiv. $b \in I(a)$) (a is $2p+2$-chained regular);
(4) $ab \sim_{r\ell}^{p+1} ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);
(5) $ab \sim_\ell \circ \sim_{r\ell}^{p} ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);

In particular, for any $p \geq 0$, if any $b \in V(a)$ is $2p$-chained regular then a is $2p$-anti-chained regular and the converse is true for $p \geq 1$ [139, Corollary 2.6]. In order to better understand these chained and anti-chained regular elements, we define inductively, for any semigroup S and any set $\Lambda \subseteq S$, $V^0(\Lambda) = \Lambda$ and

$$V^{p+1}(\Lambda) = V\left(V^p(\Lambda)\right) = \bigcup_{b \in V^p(\Lambda)} V(b).$$

(In case of a single element, we write $V^p(a)$ instead of $V^p(\{a\})$). By induction, the following equality also holds:

$$V^{p+1}(\Lambda) = V^p\left(V(\Lambda)\right) = \bigcup_{b \in V(\Lambda)} V^p(b).$$

We now characterize $2p+2$-chained regular elements in terms of $V^p(S^\#)$.

Proposition 16.1.2 ([139, Proposition 2.7]). Let S be a semigroup, $a \in \text{reg}(S)$ and $p \in \mathbb{N}$. Then the following statements are equivalent:
(1) a is $2p+2$-chained regular (for all $b \in V(a)$, $ab \sim_{r\ell}^{p+1} ba$);
(2) $V^p(a) \cap S^\# \neq \emptyset$;
(3) $a \in V^p(S^\#)$.
In particular, S is $2p+2$-chained iff $reg(S) = V^p(S^\#)$.

In the particular case $p = 0$, this allows to identify 2-chained regular elements with completely regular (equiv. group invertible) elements.

16.2) Some special cases: 1 and 2-chains

16.2.1) 1-chains

Proposition 16.2.1 ([139, Proposition 4.1]). Let S be a semigroup and $a \in S$. Then the following statements are equivalent:
(1) a is 1-anti-chained regular (resp. 1-chained regular);
(2) a is regular and $a = a^2b, b = b^2a$ (resp. $a = ba^2, b = ab^2$) for all $b \in V(a)$;
(3) a is completely regular and $ab = aa^\#$ (resp. $ba = aa^\#$) for all $b \in V(a)$.

Corollary 16.2.2 ([139, Proposition 4.2]). A semigroup S is left (resp. right, resp. both) 1-chained iff isomorphic idempotents are $\sim_\ell$-related (resp. $\sim_r$-related, resp. equal) iff $\mathrm{reg}(S) = S^\#$ and for all $a \in \mathrm{reg}(S)$ and any $b \in V(a)$, $ab = aa^\#$ (resp. $ba = a^\# a$, resp. $b = a^\#$).
In particular, 1-chained semigroups and 0-chained semigroups coincide.

The case of weakly 1-chained semigroup does not appear in [139], but it is not difficult to prove that such semigroups have $\mathcal{D}$-classes either left or right 1-chained.

In case of a regular semigroup, we deduce directly that a semigroup S is regular and (left and right) 1-chained iff it is completely regular and inverse iff it is regular semigroup with central idempotents (a *Clifford semigroup*) iff it is a semilattice of groups. However, without regularity, a (left and right) 1-chained semigroup need not have central idempotents.

Example 16.2.1 ([139, Example 4.3]). Let $S = \langle e, a | e^2 = e \rangle$, quotient of the free semigroup with two generators e, a by the relation $e^2 = e$. Then e is the only idempotent hence S is 1-chained. But $ea \neq ae$.

On the other hand, consider the bicyclic semigroup $\mathcal{M} = \{< p, q > / pq = 1\}$. It is *bisimple* (it has a single $\mathcal{D}$-class) and *inverse* ($\mathcal{M}$ is regular and idempotents commute). However, $pq = 1$ and qp are $\mathcal{D}$-related hence isomorphic but neither left nor right associates. (More generally, any n-chained monoid $\mathcal{M}$ is Dedekind-finite: $(\forall p, q \in \mathcal{M})\ pq = 1 \Rightarrow qp = 1$).

The conclusion is much more stronger for rings, for with or without regularity, idempotents are central. In fact, a ring R is 1-chained iff R is an abelian ring (idempotents are central) [139, Theorem 4.4].

Going back to the semigroup case, it happens that a general structure theorem still holds under replacement of regularity by π-regularity (a semigroup S is *(completely) π-regular* if each element of S has a power which is (completely) regular). In [20]*, Bogdanović et al. study *uniformly-π-inverse* semigroups, that are π-regular semigroups with the additional assumption that $axa = a$ implies $ax = xa$, and prove that their structure is perfectly known [20, Theorem 5.10]*. By Corollary 16.2.2, these semigroups are precisely the π-regular 1-chained semigroups. We need some terminology: a semigroup S is *Archimedean* (resp. *t-Archimedean*) if for any $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in S^1 b S^1$ (resp. $a^n \in S^1 b \cap b S^1$). S is *completely Archimedean* if it is Archimedean and completely π-regular.

Corollary 16.2.3 (from [20, Theorem 5.10]*). Let S be a semigroup. Then the following statements are equivalent:
(1) S is π-regular and 1-chained;
(2) S is uniformly-π-inverse;
(3) S is π-regular and a semilattice of t-Archimedean semigroups;
(4) S is a semilattice of nil-extensions of groups.

16.2.2) 2-chains

From Proposition 16.1.2, a is 2-chained regular iff it is completely regular. And by exchanging the role of the idempotents in the definition of left 2-chained semigroups, left and right 2-chained semigroups coincide. Thus [139, Corollary 4.10] a semigroup is left (equiv. right, equiv. both) 2-chained iff regular elements are completely regular (in particular, the regular and 2-chained semigroups are the completely regular ones, whose structure is well-known, see Chapter 2). At the present time, there are to my knowledge no structure theorems for non-regular 2-chained semigroups in full generality. On the other hand, as for 1-chained semigroups much can be said under the additional assumption that semigroup S is also π-regular. Indeed, [20]*is precisely a survey article (with many references therein) on *uniformly π-regular rings* (the uniformly-π-inverse rings being just a special case), which are explicitly defined as π-regular rings in which every regular element is strongly regular (and as such they are in particular strongly π-regular rings), and the semigroup case is also studied.

Corollary 16.2.4 (From [20, Theorem 5.7]*)**.** Let S a semigroup. Then the following statements are equivalent:
(1) S is π-regular and 2-chained;
(2) S is uniformly π-regular (S is π-regular and every regular element is completely regular);
(3) S is completely π-regular and a semilattice of Archimedean semigroups;
(4) S is a semilattice of completely Archimedean semigroups.

Once again the case of rings is very specific. From [115, Theorem 3.13]*, a ring R is 2-chained iff its is weakly 2-chained iff idempotents are central modulo the Jacobson radical.

As explained, for the moment, the structure of non-regular 2-chained semigroups (let alone weakly 2-chained semigroups and 3-chained semigroups) is unknown. This seems a very challenhing task, as may be the study of weakly 2-chained semigroups or 3-chained semigroups under regularity or π-regularity assumptions.

Chapter 17

Semigroup biacts

In [152], we describe a class of semigroup biacts that is analogous to the class of completely simple semigroups, and provide the reader with structure theorems for those biacts that is analogous to the Rees-Sushkevitch Theorem. These theorems describe stable, $\mathcal{J}$-simple biacts in terms of wreath products, translations of completely simple semigroups, biacts over endomorphism monoids of free G-acts, tensor products and matrix biacts. Applications to coproducts and left acts are also given.

Since most of the constructions of the paper are rather technical, we refer to [152] for the precise definitions and statements. Still, many new notions have to be defined. In this memoir, we first present semigroup biacts and their associated category, and Green's relations upon them. Then we discuss some of the principal results, under their simplest form.

17.1) Prerequisites on semigroup biacts

We will need some definitions and notations. In this chapter, in order to distinguish between a semigroup S and its underlying set, we will denote the latter by $\underline{S}$ (so that formally $S = (\underline{S}, .)$).

17.1.1) Semigroup biacts, and their category

A *right semigroup act* is a triple $\mathbf{X} = (X, S, \beta)$ where X is a set, S is a semigroup, and $\beta : X \times S \to X$ is a *semigroup action*, that is, a function such that for all $s, s' \in S$ and $x \in X$ $\beta(x, ss') = \beta(\beta(x, s), s')$. Left semigroup acts are defined dually.

By a *semigroup biact*, we mean a 5-tuple $\mathbf{X} = (T, X, S, \alpha, \beta)$ where (T, X, α) and (X, S, β) are left and right semigroup acts and the following compatibility condition holds:

$$(\forall t \in T, \forall x \in X, \forall s \in S)\ \alpha(t, \beta(x, s)) = \beta(\alpha(t, x), s).$$

For any $t \in T, x \in X$ and $s \in S$, when no confusion is possible, we will simply denote $\alpha(t,x)$ by tx (or $t \cdot x$) and $\beta(x,s)$ by xs (or $x \odot s$) and simply refer to the biact as the triple $\mathbf{X} = (T, X, S)$. The compatibility condition then reads

$$(\forall t \in T, \forall x \in X, \forall s \in S)\ t(xs) = (tx)s$$

and the expression $txs = t(xs) = (tx)s$ is unambiguous. Semigroup biacts for a category **SemBiact** where:

- Objects are semigroups biacts $\mathbf{X} = (T, X, S)$;
- Morphisms $(T, X, S) \to (T', X', S')$ are triples $\Phi = (\phi, f, \psi)$ where $f : X \to X'$ is a function, $\phi : T \to T'$ and $\psi : S \to S^{'}$ are semigroup morphisms and

$$(\forall t \in T, \forall s \in S, \forall x \in X)\ f(tx) = \phi(t)f(x),\ f(xs) = f(x)\psi(s).$$

The categories **LeftSemAct** and **RightSemAct** are defined accordingly. Isomorphisms and embeddings in our forthcoming structure theorems are understood in these categories.

The preference of semigroup biacts over semigroup acts is for reasons of symmetry and duality, that will become more obvious in the study of *Green's relations* on biacts.

We now provide the reader with three examples of monoid biacts that will prove useful in some of our structure theorems.

Example 17.1.1 ([152, Example 3.11, Examples 2.1 and 2.2])**.**

(1) Let S be a semigroup. Then $(S, \underline{S}, S)$ is a semigroup biact.

(2) Let S be a semigroup and e, f two idempotents of S. Then $(eSe, e\underline{S}f, fSf)$ is a semigroup (actually a monoid) biact.

(3) The formal construction (2) specializes to the construction of certain matrix biacts. Let R be a ring. If we set $S = M_{p+q,p+q}(R)$, $e = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 0 & I_q \end{pmatrix}$, then we can identify $M_{p,p}(R)$, $M_{q,q}(R)$ (as monoids) and $M_{p,q}(R)$ (as a set) with the upper left corner $eSe = \begin{pmatrix} M_{p,p}(R) & 0 \\ 0 & 0 \end{pmatrix}$, down right corner $fSf = \begin{pmatrix} 0 & 0 \\ 0 & M_{q,q}(R) \end{pmatrix}$ and upper right corner $e\underline{S}f = \begin{pmatrix} 0 & M_{p,q}(R) \\ 0 & 0 \end{pmatrix}$ respectively. We obtain a semigroup biact of matrices:

$$\boldsymbol{R_{p,q}} = (M_{p,p}(R), M_{p,q}(R), M_{q,q}(R))$$

with biaction the matrix product.

17.1.2) The regular representation, fatihful biacts, endomorphims

Let X be a set, $\mathcal{T}(X)$ be the set of transformation on X and T be a subsemigroup of $\mathcal{T}(X)$. Then T acts on X on the left by evaluation, so that any transformation semigroup defines a left semigroup act. Conversely, a classical argument associates to any left act (T, X) a subsemigroup of $\mathcal{T}(X)$ that is a homomorphic image of T. The morphism $\phi : T \to \mathcal{T}(X)$ is defined by $\phi(t) = \delta_t$ for all $t \in T$, where $\delta_t : x \mapsto tx$ is the left translation on X induced by t. This construction extends to biacts as follows.

Two subsemigroups $T \trianglelefteq \mathcal{T}(X)$ and $S \trianglelefteq \mathcal{T}^{op}(X)$ are *compatible* if they commute as functions from X to X, that is for any $x \in X, f \in T$ and $g \in S$ we have that $(f(x))\, g = f\,((x)g)$. If this is the case, they define a semigroup biact (T, X, S).

Conversely, let $\mathbf{X} = (T, X, S)$ be an object in **SemBiact**. For any $t \in T$ one can define the left translation $\delta_t \in \mathcal{T}(X)$ by $\delta_t : x \mapsto tx$ (resp. for any $s \in S$, the right translation $\tau_s \in \mathcal{T}^{op}(X)$ by $\tau_s : x \mapsto xs$). Then $\phi : T \to \mathcal{T}(X)$, $t \mapsto \delta_t$ is a semigroup homomorphism from T to $\mathcal{T}(X)$ and dually, $\psi : S \to \mathcal{T}^{op}(X)$, $s \mapsto \tau_s$ is a semigroup homomorphism from S to $\mathcal{T}^{op}(X)$, such that $\phi(T)$ and $\psi(S)$ are compatible. Putting $RegT = \phi(T)$ and $Reg(S) = \psi(S)$ we have the biact $\mathbf{RegX} = (RegT, X, RegS)$ is the *regular representation* of $\mathbf{X} = (T, X, S)$.

If $\Phi = (\phi, id_X, \psi)$ is an isomorphism then we say that T and S act faithfully on X, or that $\mathbf{X} = (T, X, S)$ is a *faithful biact*.

This representation by functions is very close to the classical case, but it can in certain cases be interestingly replaced by the following one. Let (T, X, S) be a semigroup biact. It makes sense to define T-*endomorphisms* and the *endomorphism monoid* $\mathrm{End}^{op}(T, X)$. Dually, we can also define $\mathrm{End}(X, S)$. As $(tx)s = t(xs)$ for all $t \in T, s \in S, x \in X$ then the right translation τ_s actually defines an element of $\mathrm{End}^{op}(T, X)$, and the left translation δ_t actually defines an element of $\mathrm{End}(X, S)$.

We therefore mostly consider $RegT$ as a submonoid of $\mathrm{End}((X, S))$ and $RegS$ as a submonoid of $\mathrm{End}^{op}((T, X))$ rather as submonoids of functions.

In particular, the following construction (inspired by the construction of the dual vector space in functional analysis) proved very useful: Let (T, X) be a left semigroup act. Then $\mathrm{End}^{op}(T, X)$ is a monoid, that acts on X on the right by point evaluation: $x \odot g = [x]g\ x \in X, g \in \mathrm{End}^{op}((T, X))$, such that $(T, X, \mathrm{End}^{op}(T, X))$ is a semigroup biact. The dual construction holds.

Lemma 17.1.1 ([152, Lemma 2.5])**.** Let (T, X, S) be a faithful biact. Then (T, X) embeds in the left act $(\mathrm{End}(X, S), X)$, and dually.

17.1.3) stable, $\mathcal{J}$-simple biacts

Analogously to Green's relations, we can define the following relations on the biact $\mathbf{X} = (T, X, S)$. These relations are defined in [118]*, but only few results are derived from these definitions. As usual S^1 (resp. T^1) denotes the monoid generated by S (resp. T). Let $x, y \in X$.

(1) $x \,\mathcal{R}\, y \iff (\exists s, s' \in S^1)\, xs = y$ and $ys' = x \iff xS^1 = yS^1$
(2) $x \,\mathcal{L}\, y \iff (\exists t, t' \in T^1)\, tx = y$ and $t'y = x \iff T^1x = T^1y$
(3) $\mathcal{H} = \mathcal{R} \wedge \mathcal{L}$
(4) $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$
(5) $x \,\mathcal{J}\, y \iff (\exists t, t' \in T^1, \exists s, s' \in S^1)\, txs = y$ and $t'ys' = x \iff T^1xS^1 = T^1yS^1$

It happens that these relations behave almost completely as their classical counterpart. In particular, they are equivalence relations on X; relation $\mathcal{R}$ (resp. $\mathcal{L}$) is a left (resp. right) congruence, and $\mathcal{R}$ and $\mathcal{L}$ commute so that $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is also an equivalence relation.

Most importantly for our purpose, Green's lemma holds for these relations.

Lemma 17.1.2 ([152, Lemma 3.5]). Let $x, y \in X$ and $s, s' \in S^1$ such that $xs = y$ and $ys' = x$ ($x \,\mathcal{R}\, y$). Then the right translation $\tau_s : z \mapsto zs$ is a bijection from L_x to L_y with inverse $\tau_{s'}$, that preserves $\mathcal{R}$-classes. In particular it sends $\mathcal{H}$-classes to $\mathcal{H}$-classes.

Let $\mathcal{K}$ denote any of these relations. Then the semigroup biact $\mathbf{X} = (T, X, S)$ is *$\mathcal{K}$-simple* if X consists of a single $\mathcal{K}$-class.

Definition 17.1.3 ([152, Definition 3.10]). A semigroup biact $\mathbf{X} = (T, X, S)$ is *left stable* (resp. *right stable, stable*) if $x \,\mathcal{J}\, tx \iff x \,\mathcal{L}\, tx$ for any $x \in X, t \in T$ (resp. $x \,\mathcal{J}\, xs \iff x \,\mathcal{R}\, xs$ for any $x \in X, s \in S$, resp. both).
It is *completely left stable* (resp. *completely right stable, completely stable*) if $x \,\mathcal{L}\, tx$ for any $x \in X, t \in T$ (resp. $x \,\mathcal{R}\, xs$ for any $x \in X, s \in S$, resp. both).

It is proved ([152, Lemmas 3.14, 3.15, 3.17]) that finite biacts are stable, that $\mathcal{J} = \mathcal{D}$ on stable biacts and that stable, $\mathcal{J}$-simple biacts are completely stable.

Also, it is proved [152, Example 3.11] that given a semigroup S, the biact $(S, \underline{S}, S)$ is completely stable iff S is completely simple.

In [152], I also describe the construction of *Schützenberger groups* of $\mathcal{H}$-classes (group of left/right translations induced by the left/right stabilizer of a $\mathcal{H}$-class H) and provides the reader with their main properties. I also prove the existence of a *coherent cross-section* of a $\mathcal{D}$-class [152, Theorem 3.23], as done by Grillet [74]*in the case of semigroups. These are actually the main technical tools needed to derive the structure of faithful, stable, $\mathcal{J}$-simple biacts.

17.2) Structure of faithful, stable, $\mathcal{J}$-simple biacts

The structure theorems of [152] exhibit stable, $\mathcal{J}$-simple biacts as (isomorphic to) certain subacts of a larger biact, whose structure is perfectly known, and defined in terms of two sets (the set I of $\mathcal{R}$-classes and the set Λ of $\mathcal{L}$-classes) and a group G (the Schützenberger group of any $\mathcal{H}$-class). This larger biact may be described (up to isomorphy) by different means detailed in [152]:

(1) wreath products [152, Theorem 4.14];

(2) translations of completely simple semigroups [152, Corollary 4.15];

(3) biacts over endomorphism monoids of free G-acts [152, Corollary 4.17];

(4) tensor products;

(5) matrix biacts [152, Corollary 4.19].

We only describe the simplest ones here, namely translations of completely simple semigroups and matrix biacts (as in Example 17.1.1). For the other constructions, we refer to [152].

Corollary 17.2.1 ([152, Corollary 4.15])**.** Let $\mathbf{X} = (T, X, S)$ be a faithful, stable, $\mathcal{J}$-simple semigroup biact. Then there exists a completely simple semigroup C, and subsemigroups T_L of $L(C)$, S_R of $R(C)$ (where $L(C)$ and $R(C)$ are the semigroups of left and right translations on C respectively) such that:
(1) $(\forall x, y \in C)\; x\,\mathcal{L}\,y$ in C iff $x\,\mathcal{L}\,y$ in $(T_L, \underline{C})$;
(2) $(\forall x, y \in C)\; x\,\mathcal{R}\,y$ in C iff $x\,\mathcal{R}\,y$ in $(\underline{C}, R_S)$;
(3) $(T, X, S) \simeq (T_L, \underline{C}, S_R)$.
Conversely, any semigroup biact of this form is faithful, stable, and $\mathcal{J}$-simple.

Before stating the matrix biact version, we need some definitions and notations. Let I, Λ be sets and G be a group. We define $\mathcal{M}^c_{I,I}(G)$ as the $I \times I$ matrices with coefficients in the monoid with zero $G \bigcup \{\star\}$ (with $\star$ the zero of the monoid) such that each column contains exactly one coefficient in G, and the others $\star$. Such matrices are sometimes called column-monomial matrices (over G). We define a partial sum on $G \cup \{\star\}$ by $\star + \star = \star$ and $\star + g = g$ for any $g \in G$, and the product on $\mathcal{M}_{I,I}(G)$ by the classical product matrix formula

$$A \times B(i, j) = \sum_{k \in I} A(i, k)B(k, j).$$

Dually, we can define $\mathcal{M}^r_{\Lambda,\Lambda}(G)$ (each row contains exactly one coefficient in G *i.e.* row-monomial matrices). And finally we define $\mathcal{M}^s_{I,\Lambda}(G)$ as the $I \times \Lambda$ matrices with coefficients in $G \cup \{\star\}$ such that exactly one coefficient is in G. Matrix multiplication (on the left and on the right) define a monoid biact

$$\boldsymbol{\mathcal{M}(I, G, \Lambda)} = (\mathcal{M}^c_{I,I}(G), \mathcal{M}_{I,\Lambda}(G), \mathcal{M}^r_{\Lambda,\Lambda}(G)).$$

Corollary 17.2.2 ([152, Corollary 4.19])**.** Let I, Λ be two sets and G a group. Let $T_I \trianglelefteq \mathcal{M}^c_{I,I}(G)$ (resp. $S_\Lambda \trianglelefteq \mathcal{M}^r_{\Lambda,\Lambda}(G)$) be a subsemigroup of the monoid of matrices over $G \cup \{\star\}$ such that for all $i, j \in I$ and all $g \in G$, there exists $M \in T_I, M(i,j) = g$ and dually for all $\lambda, \mu \in \Lambda$ and all $g \in G$, there exists $M \in S_\Lambda, M(\lambda, \mu) = g$. Then the biact $(T_I, \mathcal{M}^s_{I,\Lambda}(G), S_\Lambda)$ is faithful, stable and $\mathcal{J}$-simple.
Conversely, any faithful, stable, $\mathcal{J}$-simple semigroup biact is isomorphic to a biact of this form.

17.3) Applications

It is worth to mention that the structure theorems obtained for faithful, stable, $\mathcal{J}$-simple semigroup biacts carry to completely stable semigroup biacts [152, Corollaries 4.23 and 4.24], completely stable left semigroup acts [152, Corollary 5.10], and faithful, $\mathcal{L}$-simple left semigroup acts [152, Corollaries 5.2 and 5.3]. In this case we recover Oehmke's main Theorem [186]*and Steinberg's version [204, Corollary 3.17]*of the Kaloujnine-Krasner Theorem [122]*. In the particular case of a faithful, $\mathcal{L}$-simple semigroup left act (T, X) with X finite, we obtain in [152, Corollary 5.7] an embedding of (T, X) in an iterated wreath product of the form $(T_p, X_p) \wr (G_p, G_p) \wr (G_{p-1}, G_{p-1}) \wr \ldots \wr (G_1, G_1)$ (where the G_i are groups, and (T_p, X_p) is a $\mathcal{L}$-simple left act such that its automorphism group $\mathrm{Aut}(T_p, X_p)$ (equiv. endomorphism monoid $\mathrm{End}(T_p, X_p)$) is trivial).

Chapter 18

Partial Orders on arbitrary (non regular) semigroups

18.1) Preliminaries on partial orders on semigroups

The first partial order on semigroups was defined on inverse semigroups by Vagner in 1952 [205]*, as the abstract counterpart of the inclusion of partial transformations in the case of the symmetric inverse semigroup. Recall that a semigroup S is an inverse semigroup if any element admits a unique reflexive inverse $a^{-1} \in V(a)$.

For $a, b \in S$, with S an inverse semigroup, Vagner defined the partial order ω by $a\omega b$ if $a^{-1}a = a^{-1}b$. Its restriction to the commutative subsemigroup of idempotents leads to the identification of commutative bands with semilattices. Actually, in this case ω restricts to the natural partial order $\leq$ on the set $E(S)$ of idempotents of S (for $e, f \in E(S), e \leq f \iff ef = fe = e$), and was therefore also called the natural partial order (on inverse semigroups).

The natural partial order on inverse semigroups was extended to to the case of regular semigroups independently by Hartwig [85]*and Nambooripad [178]*in 1980. At this time, regular semigroups occupied already a prominent place within semigroup theory. This order was later extended by Mitsch to non-regular semigroups [171]*. Restricted to idempotents, the defined relation once again reduces to the natural partial order. In [171]*, it is proved that the three relations indeed coincide on a regular semigroups.

Lemma 18.1.1 ([171, Lemma 1]*)**.** For a regular semigroup S, the following conditions are equivalent:
(1) $a = eb = bf$ for some $e, f \in E(S)$;
(2) $a = aa'b = ba''a$ for some $a', a'' \in V(a)$ (equiv. $a' \in I(a)$);
(3) $a = aa'b = ba'a$ for some $a' \in V(a)$ (equiv. $a' \in I(a)$);
(4) $a'a = a'b$ and $aa' = ba'$ for some $a' \in V(a)$ (equiv. $a' \in I(a)$) (Hartwig [85]*);
(5) $a = ab'b = bb'a, a = ab'a$ for some $b' \in V(b)$;
(6) $a = axb = bxa, a = axa$ for some $x \in S$;
(7) $a = axb = bxa, a = axa, b = bxb$ for some $x \in S$;
(8) $a = eb$ for some idempotent $e \in R_a$ and $aS \subseteq bS$ (Nambooripad [178]*);
(9) $a = xb = by$, $xa = a$ for some $x, y \in S^1$ (Mitsch [171]*);
(10) $a = bzb$ for some $z \in S^1$ and $I(b) \subseteq I(a)$ (Hartwig and Luh, see [167]*).

These statements define relations on any (non necessarily regular) semigroup, but they may then fail to be equivalent. Also, in the non regular case these relations may fail to be reflexive. Therefore, as done in [78] and [79], we adopt in this chapter the convention that a partial order is an antisymmetric and transitive relation only (no reflexivity required).

Relations (4) and (6) are equivalent on any semigroup [79, Lemma 1]. They are called the Hartwig-Nambooripad order afterward, and denoted by $<_{\mathcal{N}}$. By $<_{\mathcal{M}}$ and $<_{\mathcal{HL}}$ we denote relations (9) and (10). We let also $\mathcal{P}$ be the relation $a\mathcal{P}b \iff a = pa = pb = bp = ap$ for some $p \in S^1$. On arbitrary semigroups, $<_{\mathcal{M}}$ and $\mathcal{P}$ are reflexive and transitive whereas $<_{\mathcal{N}}$ is transitive but fails to be reflexive on non-regular semigroups.

The purpose of [79] was threefold: first, to provide equivalent characterizations for the Hartwig-Nambooripad order based on outer inverses; second to define new partial orders on arbitrary semigroups; and third to consider them in the particular case of *epigroups* (or *group-bound*, any element admits a power that is group invertible).

18.2) Use of outer inverses, and new partial orders

In [78] and [79], we prove that the Hartwig-Nambooripad order can be defined by means of outer inverses.

Proposition 18.2.1 ([78, Lemma 3.2] and [79, Proposition 1])**.** Let $a, b \in S$. Then the following statements are equivalent:

1. $a = bxb$ for some $x \in W(b)$;
2. $a = axa = axb = bxa$ for some $x \in W(b)$;
3. $a = axa = axb = bya$ for some $x, y \in W(b)$;
4. $a = axa = axb = bya$ for some $x, y \in S$;
5. $a <_{\mathcal{N}} b$ ($a = axa = axb = bxa$ for some $x \in S$).

As by definition, for any $a, b \in S$, $a <_{\mathcal{N}} b$ implies that a is regular, relation $<_{\mathcal{N}}$ is not well suited to compare non-regular elements. Therefore, we propose in [79] the following definition, that extends the one in [78].

Definition 18.2.2 ([79, Definitions 6 and 7]). For any $a, b \in S$, let:
(1) $a\Gamma b$ if there exist $x, y \in S^1$ such that $a = axb = bya$ and $I(b) \subseteq I(a)$;
(2) If b is not regular, then $a\Gamma_l b$ (resp. $a\Gamma_r b$, $a\Gamma_{\mathcal{P}} b$) iff there exists $x \in S^1$ such that $a = axb$ (resp. there exists $y \in S^1$ such that $a = bya$, there exist $x \in S^1$ such that $a = axb = bxa$);
(3) If b is regular, then $a\Gamma_l b$ (resp. $a\Gamma_r b$, $a\Gamma_{\mathcal{P}} b$) iff there exist $x, y \in S^1$, such that $a = axa = axb = bya$ (resp. there exist $x, y \in S^1$, such that $a = aya = axb = bya$, there exists $x \in S^1$, such that $a = axa = axb = bxa$).

Lemma 18.2.3 ([79, Lemma 4 and Corollary 1]). (1) $\Gamma = \Gamma_r \cap \Gamma_l \subseteq <_{\mathcal{HL}}$;
(2) for any $a, b, \in S$, $a\Gamma_{\mathcal{P}} b$ iff $a = axb = bxa$ for some $x \in S^1$ and $I(b) \subseteq I(a)$.

And in case b is regular, we proved that [79, Lemma 4 and Corollary 1]:

$$a\Gamma_l b \iff a\Gamma_r b \iff a\Gamma b \iff a\Gamma_{\mathcal{P}} b \iff a <_{\mathcal{N}} b$$

and this is also equivalent with

$$a = ab'b = bb'a = ab'a \text{ for some } b' \in V(b).$$

In particular, on regular semigroups all four relations $\Gamma, \Gamma_r, \Gamma_l$ and $\Gamma_{\mathcal{P}}$ of Definition 18.2.2 coincide with Mitsch partial order $<_{\mathcal{M}}$.

The main property of these relations is that they all remain partial orders on arbitrary semigroups, and so is $<_{\mathcal{HL}}$.

Lemma 18.2.4 ([79, Lemmas 6 and 7 and Corollary 3]). $\Gamma, \Gamma_r, \Gamma_l, \Gamma_{\mathcal{P}}$ and $<_{\mathcal{HL}}$ are partial orders.

Also, [79] provides examples that these relations are distinct, and also distinct from $\mathcal{P}$ and $<_{\mathcal{M}}$ in general.

To conclude this section, we consider characterizations of these new partial orders in terms of outer inverses. This is the content of next two results. The first one -Corollary 18.2.5- considers $a\Gamma b$ with a regular, whereas in the second one -Proposition 18.2.6- a is arbitrary, but the semigroup S is *group-bound* (or an *epigroup*: any element has a power completely regular, or equivalently admits a Drazin inverse).

Corollary 18.2.5 ([79, Corollary 5]). Let $a, b \in S$ such that a is regular. Then the following statements are equivalent.

1. $a <_{\mathcal{HL}} b$;
2. $a = axb = bya$ for some $x, y \in W(b)$;
3. $a\Gamma b$.

If b is regular, this is moreover equivalent to $a = axa = axb = bxa$ for some $x \in W(b)$.

A one-sided version (for Γ_r and Γ_l) exists [79, Proposition 2]. In the case of an epigroup, the characterization (2) of relation Γ remains valid [79, Proposition 3].

Proposition 18.2.6 (protect[79, Proposition 3])**.** Let S be an epigroup, and $a, b \in S$. Then the following statements are equivalent.

1. $a\Gamma b$;
2. $a = axb = bya$ for some $x, y \in W(b)$.

Conclusion, open problems and future work

The previous chapters illustrate my contribution to the general algebraic theory of semigroups, notably through the use of generalized inverses and (extended) Green's relations. The spectrum covered is very large, but my contribution is very small compared to the new questions that arise. Here is a selection of a few problems I would like to address in the future:

(1) The Schützenberger category of a semigroup is a very nice tool to study a specific semigroup. I am currently trying to use them globally to study the category of semigroups itself, notably in order to interpret Morita equivalence of semigroups;

(2) Many scholars have studied the lattice of subvarieties of the variety $\mathcal{CR}$ of completely regular (unary) semigroups (see [193]*and the references therein). This should serve as an inspiration for a global study of the lattice of varieties of (completely) $\left(S^+, \widetilde{\mathcal{H}}_{S^+}\right)$-abundant semigroups;

(3) On the same topic, but regarding these semigroups as plain semigroups, one may ask the following question: given a semigroup S, is there a distinguished set E of idempotents for which S is completely $\left(E, \widetilde{\mathcal{H}}_E\right)$-abundant? If moreover we are given some semilattice decomposition (for instance the greatest one), then by Theorem 15.2.4 the set E is necessarily the union of the maximal idempotents in each class, and we have to check that each class is completely E-simple, and a type Π property (given by Theorem 15.2.2);

(4) Another very interesting challenge is the study of chained semigroups: non-regular (strongly or weakly) 2-chained semigroups, weakly 2-chained regular (or π-regular) semigroups, strongly 3-chained (non-regular, regular or π-regular) semigroups... For the moment, only local theorems (on the structure of $\mathcal{J}$-classes) seem to be attainable. Also, 3-chained rings are endomorphism rings of *perspective modules* (isomorphic direct summands always have of common complementary summand). Is there a similar theory involving S-acts, their endomorphism monoid and properties of certain subacts?

(5) The structure of semigroup acts/biacts is little developed. It would be very stimulating to find other interesting classes of biacts that admit nice structure theorems.

V

Applications of generalized inverses and idempotents to ring and module theory

Part V - Applications of generalized inverses and idempotents to ring and module theory

In this part, I present the results of my research specific to ring and module theory. They are of different flavor, depending on their topic (some works mixing different topics). However, some general scheme of research can be given.

- I usually prefer to start my studies at the level of the elements of the ring. Then, but only in a second time, would I use this element-wise approach to obtain global results at the level of the ring. In a third time, this sometimes leads to module-theoretical results by considering the ring of endomorphisms of the module;
- I try to avoid the use of an identity (when possible), so that the results would also apply to general rings;
- I try to use the additive structure scarcely, making semigroup proofs (notably proofs based on generalized inverses) when possible.

The subjects studied are:

1. The characterization of specific elements of a ring (such as clean or exchange elements) by generalized inverses, in particular the group inverse, the inverse along an element and the (b, c)-inverse (Chapter 20);
2. The study of formulas (Reverse order law, Cline's formula, Jacobson lemma) for these generalized inverses (Section 21);
3. The extension of the previous notions to general (non-unital) rings (Chapter 22);
4. The study of chains of associated idempotents, and their relation to perspectivity of submodules, group invertible elements, special clean elements, but also arithmetic and "Euclid's algorithm" (Chapter 24).

Chapter 19

Prerequisites and known results

Let R be a ring. As usual we denote by $E(R)$ (or $\text{idem}(R)$) its set of idempotents and by R^{-1} (or $\text{U}(R)$) its set of units (invertible elements). We let also $N(R)$ denote its set of nilpotent elements and $J(R)$ denote its Jacobson radical. For any idempotent $e \in E(R)$, $\bar{e} = 1-e \in E(R)$ denotes its complementary idempotent. In case of a general (a.k.a. possibly non-unital) ring, we will preferably use the notation $\Re$ to denote this general ring.

In this part, the inverse along an element and the (b,c)-inverse appear only with respect to idempotents. For the readers that do not want to dwell into the whole theory, as exposed in part II, one can take the following equalities as defining notations. Let $a \in R$ and $e, f \in E(R)$.

- the inverse of a along e is the genuine inverse $(eae)^{-1}$ of eae in the corner ring eRe, that is $a^{-e} = (eae)^{-1}_{eRe}$ (if it exists). It is called the *Bott-Duffin inverse* of a relative to the idempotent e by Khurana et.al. [113, Definition 2.12];
- the (e,f)-inverse of a is the unique element $x \in eRf$ such that $x(fae) = e, (fae)x = f$ (if it exists).

Due to their ubiquity in mathematics, some *a posteriori* equivalent notions studied in this part have been studied under different names and with *a priori* distinct definitions depending on the context (ring theory, semigroup theory, functional analysis,...) In this first section, I recall the principal notions and their equivalences.

An element $a \in R$ is:

1. regular (resp. unit regular) if $a \in aRa$ (resp. $a \in aR^{-1}a$);
2. strongly regular if $a \in a^2R \cap Ra^2$;
3. strongly π-regular if $a^n \in a^{n+1}R \cap Ra^{n+1}$ for some integer n;
4. simply polar if $ap = pa = 0$, $a + p \in R^{-1}$ for some $p \in E(R)$ (p is called the spectral idempotent or spectral projection of a);

5. polar if exists $p \in E(R)$, $ap = pa \in N(R)$, $a + p \in R^{-1}$;

6. inner invertible (resp. outer or weakly invertible) if $a = axa$ for some inner inverse $x \in R$ (resp. $x = xax$ for some outer or weak inverse $x \in R$);

7. group invertible if exists $x \in R, xa = ax, axa = a, xax = x$. In this case x is unique, called the group inverse of a and denoted by $a^{\#}$;

8. Drazin invertible if exists $x \in R, xa = ax, x^2a = x, xa^{n+1} = a^n$ for some integer n (equivalently, $xa = ax$, $x^2a = x$ and $a^2x - a \in N(R)$ by [121] Proposition 4.9). Such x is also unique if it exists, called the Drazin inverse of a and denoted by a^D;

9. strongly Drazin invertible if exists $x \in R, xa = ax, x^2a = x, a - ax \in N(R)$;

10. clean if $a = e + u$ for some idempotent $e \in E(R)$ and unit $u \in R^{-1}$ (in some papers and results, we preferably write the clean decomposition $a = \bar{e} + u$);

11. special clean if $a = e+u$ for some $e \in E(R)$ and $u \in R^{-1}$ such that $aR \cap eR = \{0\}$;

12. strongly clean if $a = e + u, eu = ue$ for some $e \in E(R)$ and $u \in R^{-1}$;

13. nil-clean if $a = e + n$ for some idempotent $e \in E(R)$ and nilpotent $n \in N(R)$;

14. strongly nil-clean if $a = e + n, ne = en$ for some $e \in E(R)$ and $n \in N(R)$;

15. left exchange (resp. right exchange, resp. exchange) if there exists $e \in E(R)$ such that $e \in Ra, 1-e \in R(1-a)$ (resp. $f \in E(R)$ such that $f \in aR, 1-f \in (1-a)R$, resp. both); Such elements are also called (left,right) suitable;

16. strongly exchange if there exists $f \in E(R)$, $f \in aR \cap Ra, 1-f \in (1-a)R \cap R(1-a)$ and $fa = af$.

A solution to $axa = a$ and $xax = x$ (both inner and outer inverse of a) is a reflexive inverse of a.

The following results are scattered in the semigroups, rings or functional analysis literature (see for instance [6]*, [27]*, [38]*, [47]*, [50]*, [58]*, [73]*, [83]*, [86]*, [111]*, [120]*, [121]*, [180]*, [182]*, [212]*). Most of them involve some commutation property.

Proposition 19.0.1. Let $a \in R$. It holds that:

1. a is unit-regular $\Leftrightarrow a = eu$ for some idempotent $e \in E(R)$ and $u \in R^{-1}$;
2. a is strongly regular $\Leftrightarrow a$ is simply polar $\Leftrightarrow a$ is group invertible $\Leftrightarrow a = eu$ for some $e \in E(R), u \in R^{-1}$ such that $eu = ue$;
3. a is strongly π-regular $\Leftrightarrow a$ is polar $\Leftrightarrow a$ is Drazin invertible;
4. a is strongly nil-clean $\Leftrightarrow a$ is strongly Drazin invertible $\Rightarrow a$ is Drazin invertible $\Rightarrow a$ is strongly clean;
5. a is left suitable iff a is right suitable iff there exists $e \in E(R)$ such that $e \in Ra, 1 - e \in (1 - a)R$;
6. a is clean $\Rightarrow a$ is exchange;
7. a is strongly exchange $\Leftrightarrow a$ is strongly clean.

Also, in the special case $\phi \in R = \mathrm{End}(M)$ where M is a right module over a ring it holds that:

(1) $M = A \oplus B = C \oplus D$ such that $\phi(A) \subseteq C, (1 - \phi)(B) \subseteq D$ and $\phi_A : A \to C, (1 - \phi)_B : B \to D$ are isomorphisms.

(2) ϕ is strongly clean $\Leftrightarrow M = A \oplus B$, $\phi = \phi_A + \phi_B$, with $\phi_A, 1_B - \phi_B$ isomorphisms;

(3) ϕ is strongly π-regular (resp. strongly regular) $\Leftrightarrow M = A \oplus B$, $\phi = \phi_A + \phi_B$ with ϕ_A an isomorphism and ϕ_B nilpotent (resp. 0);

(4) ϕ is strongly nil-clean $\Leftrightarrow M = A \oplus B$, $\phi = \phi_A + \phi_B$, with $1_A - \phi_A, \phi_B$ both nilpotent.

(In the three latter cases, $\phi(A) \subseteq A, \phi(B) \subseteq B$ and by ϕ_A, ϕ_B we mean $\phi_A : A \to A, \phi_B : B \to B$).

Finally, if $T \in R = L(X)$ space of bounded linear operators on a Banach space X then T is polar (resp. simply polar) with $p \neq 0 \Leftrightarrow 0$ is a pole (resp. simple pole) of the resolvent (and in this case p is the spectral projection on the spectral set $\{0\}$) $\Leftrightarrow X$ is a topological direct sum $X = M \oplus N$ with $M \neq 0$, $T_M : M \to M$ nilpotent (resp. 0) and $T_N : N \to N$ invertible.

Chapter 20

Exchange elements, (special) clean elements and generalized inverses

In order to to study refinements of direct sum decompositions (as does Schreier's theorem for groups), Crawley and Jònsson [44]*introduced the exchange property (an analog of Steinitz' Exchange Lemma for vector spaces). Based on that property, Warfield [214]*studied "Krull-Schmidt decomposition theorems". Two direct decompositions $A = M \oplus N = \bigoplus_{i \in I} A_i$ of a module A can be exchanged at M if we can refine the direct summands A_i to submodules $A_i' \subseteq A_i, i \in I$ such that $A = M \oplus \left(\bigoplus_{i \in I} A_i'\right)$. The module M is said to have the *n-exchange property* (with n a cardinal) if any pair of decompositions $A = M' \oplus N = \bigoplus_{i \in I} A_i$ with $M \simeq M'$ can be exchanged at M' for any $\mathrm{card}(I) \leq n$. The (finite) exchange property is the n-exchange property for all (finite) cardinals n. For a modern presentation on this topic, see [64]*.

It was later observed by Warfield [213]*that the exchange property is an "ER"-property (endomorphism ring property as coined by T-Y. Lam), in that it depends only on the endomorphism ring $R = \mathrm{End}(M)$ of the module. In [181]*, Nicholson defined an element-wise analog of an exchange ring. An element is *left suitable* (or *left exchange*) if there exists $e \in E(R)$ such that $e \in Ra, 1 - e \in (1 - a)R$. Then he proved that a ring is exchange iff all its elements are left (equiv. right) suitable.

The clean property was first defined by Nicholson [180]*as a refinement of the exchange property. Indeed, he observed that most exchange rings shared this stronger property, which was nicer to handle. As precisely defined in [182]*, an element $a \in R$ is clean if $a = e + u$ for some $e \in E(R)$ and $u \in R^{-1}$.

Apparently, the use of the additive law in both definitions of exchange and clean elements divert us from the purely multiplicative world of generalized inverses, but a study of exchange and clean properties via generalized inverses is still possible, as proved in [141], [153] and [161]. To my opinion, this offers at least three main advantages:

- it makes clearer certain implications/set inclusions, in particular the following ones: special clean elements are unit-regular, strongly regular elements are strongly clean, clean elements are exchange;

- characterizations in terms of inverses along a commuting idempotent offers the possibility to derive Cline's formula and Jacobson lemma;
- it opens the possibility to work with general rings.

20.1) Exchange (or suitable) elements by outer inverses

The fact that exchange elements can be characterized by outer inverses seems to be folklore, but the only reference I found in the literature is the slightly different (module-theoretic) in [172, Theorem 1]*. Therefore I stated the following result with a proof in [153].

Proposition 20.1.1 ([153, Proposition 2.2])**.** Let $a \in R$. Then the following statements are equivalent:
(1) there exists $x, y \in R$ such that:
 (a) $xy = xay$;
 (b) $(1-a)y = 1 - ax$.
(2) a is exchange;
(3) there exists $x, y \in R$ such that:
 (a) $xax = x$;
 (b) $y(1-a)y = y$;
 (c) $(1-a)y = 1 - ax$;

Note that a couple (x, y) solution to $(1) : xy = xay$ and $(1-a)y = 1-ax$ needs not also solve $y(1-a)y = y$. Take for instance $a = x = 1$ and any $y \neq 0$. Note also that a couple (x, y) solution to (3) satisfies $yax = 0$, a condition that can replace $(1.a)$: $xy = xay$ in the proposition. And finally a dual characterization (with $y(1-a) = 1-xa$ instead of $(1-a)y = 1-ax$) also holds.

Using [111, Theorem 3.2]*, we can actually prove a slightly different result.

Proposition 20.1.2 (unpublished)**.** Let $a \in R$. Then the following statements are equivalent:

1. there exists $x, y \in R$ such that:
   (a) $xy = 0$;
   (b) $y(1-a) = 1 - ax$.
2. a is exchange;
3. there exists $x, y \in R$ such that:
   (a) $xax = x$;
   (b) $y(1-a)y = y$;
   (c) $y(1-a) = 1 - ax$.

Proof.

(1) ⇒ (2) Assume that there exists $x, y \in R$ that satisfy (1) and pose $e = ax$. Then $e \in aR$ and $1 - e \in R(1-a)$. We compute $e - e^2 = ax(1-ax) = axy(1-a) = 0$, so that $e \in E(R)$ and a is exchange.

(2) ⇒ (3) Assume that a is exchange, with $e \in E(R) \cap aR$, $1-e \in R(1-a)$ (by [111, Theorem 3.2]*). Then there exists $u, v \in R$ such that $f = au, \bar{f} = 1 - f = v(1-a)$. Pose $x = uf, y = \bar{f}v$. Then $xax = ufauf = uf^3 = uf = x$, and symmetrically $y(1-a)y = \bar{f}v(1-a)\bar{f}v = \bar{f}^3v = y$. Also $y(1-a) = \bar{f}v(1-a) = \bar{f}^2 = \bar{f} = 1 - f = 1 - ax$.

(3) ⇒ (1) As $x = xax$, $y(1-a)y = y$ and $y(1-a) = 1 - ax$ then $xy(1-a) = x - xax = 0$ so that $xy = xy(1-a)y = 0$.

□

20.2) Clean elements by generalized (outer) inverses

In the rest of the part, we will preferably express the clean decomposition as $a = \bar{e} + u$ ($e \in E(R), u \in R^{-1}$).

The main theorem of [153] expresses cleanness in terms of (e, f)-inverses, with e, f idempotents (Bott-Duffin (e, f)-inverses).

Theorem 20.2.1 ([153, Theorem 2.1])**.** Let $a \in R$. Then a is clean if and only if exist idempotents $e, f \in E(R)$ such that:

1. $\bar{f}ae = 0$ $(fae = ae)$ and $f(1-a)\bar{e} = 0$ $(\bar{f}(1-a)\bar{e} = (1-a)\bar{e})$;
2. a has a (e, f)-inverse and $1 - a$ has a $(\bar{e}, \bar{f})$-inverse.

In this case $a = \bar{e} + u$ with $u^{-1} = a^{-(e,f)} - (1-a)^{-(\bar{e},\bar{f})}$, and e, f are similar with $f = ueu^{-1}$.

Obviously, we can carry a dual construction (and get a second idempotent $g = u^{-1}eu$). Equivalently, we can work with principal ideals.

Corollary 20.2.2 ([153, Corollary 2.1])**.** Let $a \in R$. Then a is clean if and only if exist idempotents $e, f \in E(R)$ such that:

1. $Re = Rae, fR = aeR$;
2. $R\bar{e} = R(1-a)\bar{e}, \bar{f}R = (1-a)\bar{e}R$.

In this case $a - \bar{e}$ is invertible.

Or we can restate Theorem 20.2.1 in terms of outer inverses only.

Corollary 20.2.3 ([153, Corollaries 2.2 and 2.3])**.** Let $a \in R$. Then a is clean if and only if exists $x, y \in R$ such that:

1. $xax = x$ (x is an outer inverse of a);
2. $y(1-a)y = y$ (y is an outer inverse of $1-a$);
3. $(1-a)y = 1 - ax$ (or its dual $y(1-a) = 1 - xa$);
4. $x - y \in R^{-1}$.

In this case $f = ax$ is an idempotent such that $Rf = Rx$ and $R\bar{f} = Ry$, and $\bar{e} = a - (x-y)^{-1}$ is an idempotent such that $eR = xR$ and $\bar{e}R = yR$. It also holds that $(1-x)R = (1-a)R$ and $(1-y)R = aR$, and that x and y are unit-regular.

Also, the invertibility condition in Corollary 20.2.3 can be replaced by a condition on principal ideals.

Proposition 20.2.4 ([153, Proposition 2.1])**.** Let $a \in R$ and assume exists $x, y \in R$ such that $xax = x$, $y(1-a)y = y$ and $(1-a)y = 1 - ax$. Then the following statements are equivalent:

1. $x - y \in R^{-1}$;
2. there exists $e \in E(R)$, $eR = xR$ and $\bar{e}R = yR$.

In this case, e is unique and satisfies $\bar{e} = a - (x-y)^{-1}$.

In the particular case of an endomorphisms ring $R = \mathrm{End}(M)$, where M is a (right) module (over a given ring k) then Theorem 20.2.1 and Corollary 20.2.2 specialize to [27, Proposition 2.2]*.

Corollary 20.2.5 ([153, Corollary 2.4])**.** Let $\phi \in R = \mathrm{End}(M)$. then ϕ is clean if and only if $M = A \oplus B = C \oplus D$, with $\phi_A : A \to C, (1-\phi)_B : B \to D$ isomorphisms.

In all the previous results, the idempotents e and f are distinct in general. We finally focus on the case $e = f$.

Theorem 20.2.6 ([153, Theorem 2.2])**.** Let $a \in R$ and assume that a is invertible along e, $1-a$ is invertible along $\bar{e} = 1-e$ for some idempotent $e \in E(R)$. Then :

1. $a^{-e} - (1-a)^{-\bar{e}}$ is invertible;
2. a is clean.

Thus the set

$$\{a \in R | a^{-e} \text{ and } (1-a)^{-\bar{e}} \text{ exist for some } e \in E(R)\}$$

defines a subset of the set of clean elements. By [153, Lemma 2.1, Theorem 5.1 and Corollary 5.1] it contains both the strongly regular and the strongly clean elements (see next section). This set will appear in our study of Cline's formula and Jacobson lemma (Chapter 21). One must keep in mind that under the previous assumptions, even if aa is clean, $a - \bar{e}$ is not a unit in general (unless for instance $ea\bar{e} = 0$ or $\bar{e}ae = 0$).

20.3) Strongly clean elements by generalized (outer) inverses

Recall that $a \in R$ is strongly clean if we can find a clean decomposition $a = \bar{e} + u, e \in E(R)$ and $u \in R^{-1}$ such that additionally, two of the three elements commute (in which case all three elements commute). Strongly clean elements are actually those clean elements for which $e = f$ in Theorem 20.2.1, for then $eae = ae$ and $e(1-a)\bar{e} = 0$, that also reads $eae = ea$.

Corollary 20.3.1 ([153, Corollary 5.1])**.** let $a \in R$, $e \in E(R)$. Then there exists $u \in R^{-1}$ such that $a = \bar{e} + u, \bar{e}u = u\bar{e}$ iff $ae = ea$, a^{-e} and $(1-a)^{-\bar{e}}$ exist, iff $ae = ea$, $e \in eaR \cap Rae$ and $\bar{e} \in \bar{e}(1-a)R \cap R(1-a)\bar{e}$.

In other words, a is strongly clean iff there exists and idempotent e commuting with a such that a is invertible along e (eae is a unit in eRe) and $1-a$ is invertible along $\bar{e}$ ($\bar{e}(1-a)\bar{e}$ is a unit in $\bar{e}R\bar{e}$).

In terms of outer inverses only we deduce the following corollary, where the additional invertibility assumption is automatic.

Corollary 20.3.2 ([153, Corollary 5.4])**.** Let $a \in R$. Then a is strongly clean if and only if there exists $x, y \in R$ such that:
1. $xax = x$, $ax = xa$;
2. $y(1-a)y = y$, $ay = ya$;
3. $(1-a)y = 1 - ax$.

In this case $x - y$ is invertible and $a = (1-ax) + (x-y)^{-1}$ is a strongly clean decomposition of a.

In particular we recover (by Proposition 20.1.1) that strongly exchange elements are strongly clean ([38, Theorem 2.2]*).

We can also combine outer inverses and ideals characterizations. We then recover one of the equivalences of [51, Theorem 5.5]*.

Corollary 20.3.3 ([153, Corollary 5.5])**.** Let $a \in R$. Then a is strongly clean if and only if exists $x \in R$ such that:
1. $xax = x$, $ax = xa$;
2. $1 - ax \in (1-ax)(1-a)R \cap R(1-a)(1-ax)$.

As noted in [51]*, in this case $(1-x)R = (1-a)R$ and $R(1-x) = R(1-a)$, which we can also recover from Corollary 20.2.3.

From the above results we get another characterization of strongly clean elements by outer inverses where:

1. the commutation is only implicit;
2. y is not assumed to be a outer inverse of $(1-a)$. Indeed, only $z = y(1-a)y$ is in general (also, $z = (1-a)^{-(1-e)}$ with $e = ax$ in this case).

Corollary 20.3.4 ([153, Corollary 5.6])**.** Let $a \in R$. Then a is strongly clean if and only if there exists $x, y \in R$ such that:

1. $xax = x$;
2. $(1-a)y = 1 - ax$ and $y(1-a) = 1 - xa$.

Using this methodology, we recover that strongly regular elements are (strongly) clean elements with an additional property (see also Section 12.1 and the commentaries of Theorem 12.1.1).

Theorem 20.3.5 ([153, Theorem 5.1])**.** Let $a \in R$. then the following statements are equivalent:

1. a is strongly regular;
2. There exist a clean decomposition $a = \bar{e} + u$ with $e \in E(R), u \in R^{-1}$ such that $a\bar{e} = 0$;
3. There exist a strongly clean decomposition $a = \bar{e} + u$ with $e \in E(R), u \in R^{-1}$ such that $a\bar{e} = 0 = \bar{e}a$.

20.4) Special clean elements by generalized (outer) inverses

Most of the results presented in this section have been previously given in Section 12.2. As explained in the foreword, they are given there a second time in order to make the part regarding ring theory self-contained.

Recall that an element $a \in R$ is *special clean* (see [1], [26]) if it admits a clean decomposition $a = \bar{e} + u$ for some $e \in E(R)$, $u \in \mathrm{U}(R)$ that satisfies the additional requirement $aR \cap \bar{e}R = \{0\}$. The set of special clean elements of R will be denoted by $\mathrm{sp.cl}(R)$.

There are thee main results. The first one ((1) $\iff$ (3) $\iff$ (4) below) expresses special cleanness as a direct sum decomposition property only. The second one ((1) $\iff$ (5)) expresses special clean elements as both clean and unit-regular elements with the same unit. And the third one ((1) $\iff$ (7)) describes the special clean elements entirely multiplicatively, as reflexive inverses of strongly regular elements (This is Theorem 12.2.1)

Theorem 20.4.1 ([153, Theorem 4.1], [141, Lemma 2.2 and Theorem 2.4], [139, Proposition 4.20])**.** Let R be a ring and $a \in R, e \in E(R)$. The following statements are equivalent:

(1) $u = a - \bar{e} \in \mathrm{U}(R)$ and $aR \cap \bar{e}R = 0$ (a is special clean);

(1′) $u = a - \bar{e} \in U(R)$ and $Ra \cap R\bar{e} = 0$;

(2) $u = a - \bar{e} \in U(R)$ and $aR \oplus \bar{e}R = R$;

(2′) $u = a - \bar{e} \in U(R)$ and $Ra \oplus R\bar{e} = R$;

(3) $aR \oplus \bar{e}R = R$ and $Ra \oplus R\bar{e} = R$;

(4) $aR \oplus \bar{e}R = R$ and $bR \oplus \bar{e}R = R$, for some $b \in V(a)$;

(5) $u = a - \bar{e} \in U(R)$ and $a = au^{-1}a$;

(6) $u = a - \bar{e} \in U(R)$, $z = u^{-1}au^{-1} \in V(a) \cap R^{\#}$ (z is a reflexive inverse of a which is strongly regular) and $zz^{\#} = e$;

(7) $aza = a, zaz = z$ and $zz^{\#} = e$ for some $z \in R^{\#}$.

Observe that the equivalence (1) $\iff$ (3) (for instance) proves the left-right symmetry of the concept of special clean element. While direct sums of right modules have been extensively studied, mixed-type decompositions (involving both right and left modules) have attracted less attention. Condition (3) claims that the direct sums conditions in (2) and (2′) together actually imply invertibility of u (hence that a is special clean). The equivalence (1) $\iff$ (4) claims that a is special clean iff aR and bR are *perspective* (share a common complementary summand) for some $b \in V(a)$. The left-right symmetry, as well as the equivalences (1) $\iff$ (5) $\iff$ (7) were also proven independently by D. Khurana, T.Y. Lam, P.P. Nielsen and J. Šter about the same time [113, Theorem 2.13]*, and are now well-known and widely used.

A very different (and probably more visual) proof of the equivalence (1) $\iff$ (6) is given in [161, Theorem 6.1] . It relies on Peirce decomposition and the following trivial fact: a group invertible element $z \in R^{\#}$ is always a unit in eRe for $e = zz^{\#}$, and conversely a unit z in some corner ring $eRe, e \in E(R)$ is always group invertible (in R).

Theorem 20.4.2 ([161, Theorem 6.1])**.** Let R be a ring and $a \in R, e \in E(R)$. Then the following statement are equivalent:

1. There exists $z \in \mathrm{U}(eRe)$ such that $aza = a, zaz = z$;
2. The Peirce decomposition of a relative to the idempotent e is of the form $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ with $a_1 \in \mathrm{U}(eRe)$ with inverse $z \in \mathrm{U}(eRe)$ and $a_4 = a_3 z a_2$;
3. $u = a - \bar{e} \in U(R)$ and $au^{-1}a = a$ (a is special clean).

Consequently, we obtain that the special clean decompositions are in bijective correspondence with completely regular reflexive inverses (already stated as Corollary 12.2.3).

Corollary 20.4.3 ([161, Corollary 6.2])**.** Let R be a ring and $a \in R$ be a special clean element. Then there is a bijective correspondence between special clean decompositions and strongly regular reflexive inverses given by $(e, u) \mapsto z = u^{-1}au^{-1}$ with reciprocal $z \mapsto (e = zz^{\#}, u = a - \bar{e})$, where $a = \bar{e} + u = au^{-1}a$ denotes the special clean decomposition.
In particular a is uniquely special clean if and only if it admits a unique reflexive inverse which is also strongly regular.

To conclude this section, we present a last (unpublished) characterization of special clean elements. It expresses special clean elements in terms of Bott-Duffin (e, f)-inverses in the spirit of Theorem 20.2.1.

Theorem 20.4.4 (unpublished)**.** Let $a \in R$. Then a is special clean if and only if exist idempotents $e, f \in E(R)$ such that:

1. $fa = a$ and $f\bar{e} = a\bar{e}$;
2. a has a (e, f)-inverse and $1 - a$ has a $(\bar{e}, \bar{f})$-inverse.

In this case $a = \bar{e} + u = au^{-1}a$ with $u^{-1} = a^{-(e,f)} - (1-a)^{-(\bar{e},\bar{f})}$, and e, f are similar with $f = ueu^{-1}$.

Proof. $\Rightarrow$ Assume that a is special clean with decomposition $a = \bar{e} + u = au^{-1}a$ for some $e \in E(R)$ and $u \in R^{-1}$. Then by Theorem 20.2.1 $f = ueu^{-1}$ satisfies that $fae = ae$, $f(1-a)\bar{e} = 0$, a has a (e, f)-inverse and $1 - a$ has a $(\bar{e}, \bar{f})$-inverse. As $f = ueu^{-1} = 1 - uau^{-1} + u$ and $au^{-1}a = a$ then $fa = a - ua + ua = a$. Then also $(f - a)\bar{e} = f(1-a)\bar{e} = 0$.

$\Leftarrow$ Assume that $fa = a$ (equiv. $\bar{f}a = 0$), $f\bar{e} = a\bar{e}$, a has a (e, f)-inverse and $1-a$ has a $(\bar{e}, \bar{f})$-inverse for some idempotents $e, f\, \mathrm{E}(R)$. Then $fae = ae$, $f(1-a)\bar{e} = 0$ so that by Theorem 20.2.1, $u = a - \bar{e}$ is invertible with inverse $u^{-1} = a^{-(e,f)} - (1-a)^{-(\bar{e},\bar{f})}$. By definition of the Bott-Duffin inverse, $aa^{-(e,f)} = faa^{-(e,f)} = f$ and $(1-a)^{-(\bar{e},\bar{f})} \in \bar{e}R\bar{f}$, so that $(1-a)^{-(\bar{e},\bar{f})}a = (1-a)^{-(\bar{e},\bar{f})}\bar{f}a = 0$. Finally

$$au^{-1}a = a\left(a^{-(e,f)} - (1-a)^{-(\bar{e},\bar{f})}\right)a = aa^{-(e,f)}a - a(1-a)^{-(\bar{e},\bar{f})}a = fa = a.$$

This proves that a is special clean.

□

Chapter 21

Reverse order law, Cline's formula and Jacobson's lemma in unital rings

In [149], the two-sided *reverse order law* (ROL) for the group inverse is studied in the general case of semigroups. Then it is proved that in a ring, it is equivalent with the one-sided ROL under Dedekind-finiteness. Precise statements are exposed in Section 21.1 (and even more precise statements including the semigroup ones in Chapter 10).

In [151], I proposed to study Cline's formula and Jacobson's lemma for weak inverses, in particular (bi)commuting ones. While Cline's formula was mostly studied in semigroups in this paper, I also proved some additional results in the ring context (independently of the existence of an identity). It it those results that are presented in the next section. Therefore, we will only deal with the two-sided results corresponding to bicommuting weak inverses. For the readers interested in the other cases, I refer to Section 4.3, or directly to [151].

Regarding Jacobson's lemma, the unital and general case differ. The study of Jacobson's lemma in general rings is thus postponed to the specific section about general rings. To be coherent with the Cline's formula case, I also present uniquely the bicommuting (hence two-sided) case, and once again refer to Section 7.4 or [151] for more general results.

21.1) Reverse order law for the group inverse in Dedekind-finite rings

Let R be a semigroup. The one-sided ROL for the group inverse is the equality $(ab)^\# = b^\# a^\#$, and the two-sided ROL the previous equality and its dual $(ba)^\# = a^\# b^\#$. These equalities are known to be false in general. Below, we first provide element-wise conditions on a and b for the two-sided ROL to hold. Second, we add an additional global finiteness condition on the ring for the one-sided ROL to hold.

Theorem 21.1.1 ([149, Theorem 2.4]). Let R be a semigroup and $a, b \in R$ be group elements. Let $a^0 = aa^\#, b^0 = bb^\#$. Then the following statements are equivalent:
(1) ab and ba are group invertible with $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$;
(2) $ab\mathcal{H}ba$;
(3) $(\exists x, y \in R)$ $ab = bxa$ and $ba = ayb$ ($ab \in bRa$ and $ba \in aRb$);
(4) $a^0 \in \{b\}'$ and $b^0 \in \{a\}'$;
(5) $a^0, b^0 \in \{a, a^\#, a^0, b, b^\#, b^0\}'$.

Recall that a ring R is *Dedekind-finite* if for any $a, b \in R$, $ab = 1$ implies $ba = 1$. By using Peirce decompositions and properties of the group inverse of triangular matrices in Dedekind-finite rings, I proved the following result.

Theorem 21.1.2 ([149, Theorem 3.16 and Corollary 3.17]). Let R be a Dedekind finite ring and $a, b \in R$ be such that a and b are group invertible. Then the following statements are equivalent:
(1) ab is group invertible with $(ab)^\# = b^\# a^\#$;
(1′) ba is group invertible with $(ba)^\# = a^\# b^\#$;
(2) ab and ba are group invertible with $(ab)^\# = b^\# a^\#$, $(ba)^\# = a^\# b^\#$;
(3) $ab\mathcal{H}ba$.

The following example shows that the one-sided ROL does not imply the two-sided ROL in general. Obviously, the ring has to be non-Dedekind finite.

Example 21.1.1 ([149, Example 3.19]). Let R be a non-Dedekind finite ring, and let $u, v \in R$ such that $uv = 1 \neq vu$. Then $(vu)^2 = vu$. Pose $w = 1 - vu$. Then $uw = wv = 0$. The ring of 3×3 matrices over R $\mathcal{M}_3(R)$ is obviously not Dedekind finite. Consider the two following matrices of $\mathcal{M}_3(R)$

$$a = \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & w & v \end{pmatrix}.$$

Then a and b are group elements with

$$a^\# = \begin{pmatrix} v & w & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix}, b^\# = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v & w \\ 0 & 0 & u \end{pmatrix}, a^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } b^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also

$$ab = \begin{pmatrix} 0 & 0 & 0 \\ 0 & vu & 0 \\ 0 & 0 & 0 \end{pmatrix} = b^\# a^\#, ba = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ w & 0 & 0 \end{pmatrix} \text{ and } a^\# b^\# = \begin{pmatrix} 0 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $ab = b^\# a^\#$ is idempotent and the reverse order law holds for ab, $(ab)^\# = b^\# a^\#$. However $ba(a^\# b^\#) \neq (a^\# b^\#)ba$ and the reverse order law does not hold for ba.

21.2) Cline's formula via lattice isomorphisms in unital rings

In this section, R is a unital ring. We are going to describe some lattices isomorphims between lattices associated to *primarily conjugate elements* in R ($u, v \in R$ are primarily conjugate if $u = ab, v = ba$ for some $a, b \in R$). As a consequence, we will derive Cline's formula for bicommuting weak inverses. In a second phase, we will extend the previous isomorphims by working on the *circle ring* R° of R.

Recall that $E(R)$ is partially ordered by $e \leq f \iff ef = fe = e$. Let C be any commutative subset of $E(R)$. Then C becomes a lattice under the two operations $e \wedge f = ef$ and $e \vee f = e + f - ef$. Moreover, this lattice is distributive. Define the *circle operation* on R as $x \circ y = x + y - xy$. Then $R^\circ = (R, \circ)$ is a monoid (with identity 0), and for any two idempotents $e, f \in C$, $e \vee f = e \circ f$.

For any $a \in R$, we define the following sets:

$$
\begin{aligned}
W_2(a) &= \{x \in R | xax = x, ca = ac \Rightarrow cx = xc (\forall c \in R)\} = W(a) \cap \{a\}'', \\
\Sigma(a) &= \{e \in E(S) | e \in aR \cap Ra\}, \\
\Sigma_2(a) &= \Sigma(a) \cap \{a\}'', \\
\Sigma^\circ(a) &= \{e \in E(S) | e \in a \circ R \cap R \circ a\}, \\
\Sigma_2^\circ(a) &= \Sigma^\circ(a) \cap \{a\}''.
\end{aligned}
$$

Lemma 21.2.1 ([151, Lemma 4.7])**.** $(\Sigma_2(a), ., \circ)$ (resp. $(\Sigma_2(a) \cap \Sigma_2^\circ(a), ., \circ)$) is a distributive lattice.

Define an operation $\odot_a$ on R by $x \odot_a y = x + y - xay$. Then

Corollary 21.2.2 ([151, Corollary 4.6])**.** Function $x \mapsto xx^\# = ax$ is an isomorphism of lattices from $(W(a) \cap \{a\}'', ._a, \odot_a)$ onto $(\Sigma(a) \cap \{a\}'', ., \circ))$. Its reciprocal maps e to a^{-e}.

And regarding conjugate idempotents we obtain:

Theorem 21.2.3 ([151, Theorem 4.4])**.** Let $u, v \in R$ be primarily conjugate elements. The lattices $\Sigma_2(u)$ (resp. $\Sigma_2(u) \cap \Sigma_2^\circ(u)$) and $\Sigma_2(v)$ (resp. $\Sigma_2(v) \cap \Sigma_2^\circ(v)$) are isomorphic. If $u = ab$, $v = ba$ then the isomorphism is given by $e \mapsto b(ab)^{-e}a$.

From the two previous results, we deduce that there is not only a bijective correspondence but a lattice isomorphism between the bicommuting weak inverses of ab and the bicommuting weak inverses of ba. We thus deduce

- Cline's formula for bicommuting weak inverses/ inverses along bicommuting idempotents -

ab is invertible along $e \in \Sigma_2(ab)$ (resp. $e \in \Sigma_2(ab) \cap \Sigma_2^\circ(ab)$) iff ba is is invertible along $f = b(ab)^{-e}a \in \Sigma_2(ab)$ (resp. $f \in \Sigma_2(ab) \cap \Sigma_2^\circ(ab)$), in which case

$$(ba)^{-f} = b\left((ab)^{-e}\right)^2 a.$$

We now push further the study in the case of rings by using twice the circle operation, and exhibit isomorphisms for a much larger class than primarily conjugate elements. The main idea is to consider the *circle ring* $R^\circ = (R, \oplus, \circ)$ with additive operation $x \oplus y = x + y - 1$. All the previous results then apply to R° and involve the ring $(R^\circ)^\circ$. But $(R, \oplus, \circ)$ is actually isomorphic to $(R, +, .)$ via the involutive map $x \mapsto 1 - x$, so that $(R^\circ)^\circ = R$. Also, we deduce from this isomorphism that $e \in \Sigma_2^\circ(a) \iff 1 - e \in \Sigma_2(1-a) \iff e \in 1 - \Sigma_2(1-a)$.

Before stating the main result, let us consider a simple case. Let u, w be primarily conjugate in R and w, v be primarily conjugate in R°, that is $u = ab$, $w = ba = d \circ c$ and $v = c \circ d$ for some $a, b, c, d \in R$. We consider Theorem 21.2.3 for both R and R° (a precise statement for R° is [151, Corollary 4.7]), and denote by $a^{\ominus e} = 1 - (1-a)^{-(1-e)}$ the inverse of a along e in R°.

(R) $e \mapsto bu^{-e}a$ is a lattice isomorphism from $(\Sigma_2(u) \cap \Sigma_2^\circ(u), ., \circ)$ onto $(\Sigma_2(w) \cap \Sigma_2^\circ(w), ., \circ)$;

(R°) $e \mapsto c \circ w^{\ominus e} \circ d$ is a lattice isomorphism from $(\Sigma_2^\circ(w) \cap \Sigma_2(w), \circ, .)$ onto $(\Sigma_2^\circ(v) \cap \Sigma_2(v), \circ, .)$.

As the opposite lattice of $(\Sigma_2^\circ(w) \cap \Sigma_2(w), \circ, .)$ is $(\Sigma_2^\circ(w) \cap \Sigma_2(w), ., \circ)$ we obtain that $e \mapsto c \circ w^{\ominus(bu^{-e}a)} \circ d$ is a lattice isomorphism from $(\Sigma_2(u) \cap \Sigma_2^\circ(u), ., \circ)$ to $(\Sigma_2(v) \cap \Sigma_2^\circ(v), ., \circ)$ [151, Corollary 4.10].

Let $\cong$ denote primarily conjugation in R and $\cong^\circ$ denote primarily conjugation in R°. Denote their transitive closure by $\equiv$ and call it *primarily equivalence*. By induction on the previous arguments we obtain that primarily equivalent elements have isomorphic lattices.

Corollary 21.2.4 ([151, Corollary 4.9])**.** Let $u, v \in R$ be primarily equivalent ($u \equiv v$). Then the lattices $(\Sigma_2(u) \cap \Sigma_2^\circ(u), ., \circ)$ and $(\Sigma_2(v) \cap \Sigma_2^\circ(v), ., \circ)$ are isomorphic.

It may be interesting to consider properties invariant by primarily equivalence rather than primarily conjugation. In this (purely ring) case, instead of the natural inverse, that corresponds to a^{-M} with M greatest element in $\Sigma_2(a)$, it seems indicated to consider instead a *binatural inverse* a^{-M}, with M greatest element in $\Sigma_2(a) \cap \Sigma_2^\circ(a)$. Corollary 21.2.4 then claims that binatural invertibility is invariant under primarily equivalence. By [147, Theorem 8] and [151, Example 4.2], if the generalized Drazin inverse exists, then the binatural inverse exists and they coincide.

Also, by [153, Theorem 2.9], if $\Sigma_2(u) \cap \Sigma_2^\circ(u)$ contains an idempotent e then $u^{-e} - (1 - u)^{-(1-e)}$ is a unit and u is clean. Thus, Corollary 21.2.4 considers non-empty lattices only for a subclass of strongly clean elements.

21.3) Jacobson's lemma in unital rings

- Jacobson's lemma in unital rings -

Corollary 21.3.1 ([151, Corollary 4.5]).
Let $e \in \Sigma_2(ab) \cap \Sigma_2^\circ(ab)$. Then $f = b(ab)^{-e}a \in \Sigma_2(ba) \cap \Sigma_2^\circ(ba)$ and

$$\begin{aligned}(1-ba)^{-(1-f)} &= 1 + b(1-ab)^{-(1-e)}a - f \\ &= 1 + b\left((1-ab)^{-(1-e)} - (ab)^{-e}\right)a.\end{aligned}$$

We have already seen that the spectral projection p of a generalized Drazin invertible element $1-ab$ satisfies that $1-p$ is the greatest element of $\Sigma_2(1-ab)$ (Theorem 7.3.2 or [147, Theorem 8]). Actually, it is proved in [151, Example 4.2] that it also holds that $p \in \Sigma_2(ab)$. Thus $y = 1 + b\left((1-ab)^{-(1-p)} - (ab)^{-p}\right)a$ seems a perfect candidate for the generalized Drazin inverse of $1-ba$. Actually, by the semilattices isomorphism properties, we already now that this is the natural inverse of $(1-ba)$, and we have only to check that $(1-ba)^2y - (1-ba)$ is quasinilpotent. This is done in [151, Example 4.3] and we recover Zhuang [229, Theorem 2.3]*formula (x^{gD} denotes the generalized Drazin inverse of $x \in R$).

$$(1-ba)^{gD} = 1 + b\left((1-ab)^{gD} - (ab)^{-p}\right)a,$$

where p is the spectral idempotent if $(1-ab)$. Also, the spectral idempotent of $(1-ba)$ is $q = b(ab)^{-p}a = b\left[p(1-p(1-ab))^{-1}\right]a$.

We also applied Cline's formula and Jacobson's lemma 21.3.1 to strongly clean and strongly nil-clean elements (a.k.a. strongly Drazin invertible elements) thanks to their characterization by means of inverses along an idempotent [153, Corollary 5.1].

Corollary 21.3.2 ([153, Corollaries 5.2 and 5.3]). Let $a, b \in R$.
(1) If ab is strongly clean so is ba. Moreover, if $ab = \bar{e} + u$ ($e \in E(R), u \in R^{-1}, eu = ue$) is a strongly clean decomposition of ab, then $ba - \bar{f}$ is invertible with $f = b(ab)^{-e}a \in E(R)$.
(2) If ab is strongly nil-clean so is ba. Moreover, if $ab = e + n$ ($e \in E(R), n \in N(R), en = ne$) is a strongly nil-clean decomposition of ab, then $ba - f$ is nilpotent with $f = b(ab)^{-e}a \in E(R)$.

It has not been done in [151], but as for Cline's formula the previous results can be understood in terms of primarily conjugate idempotents, and then extended to primary equivalent idempotents.

Corollary 21.3.3 (unpublished). Let $u, v \in R$ be primary equivalent idempotents. Then there is a bijective correspondence between the bicommuting outer inverses of $1-u$ of the form $(1-u)^{1-e}, e \in \Sigma_2(u) \cap \Sigma_2^\circ(u)$ and the bicommuting outer inverses of $1-v$ of the form $(1-v)^{1-f}, f \in \Sigma_2(v) \cap \Sigma_2^\circ(v)$.

Chapter 22

From unital to general rings

In order to deal with general rings, it has long been noticed that a interesting tool is the so-called *circle operation* $x \circ y = x + y - xy$ [4]*, [95]*, [101]*, [124]*, [134]*. This operation is associative, and that if R is a unital ring, then $x \mapsto 1 - x$ is an involutive isomorphism of monoids from $(R, .)$ onto $(R, \circ)$. In the case of a general ring $\Re = (\Re, +, .)$ $\Re^\circ = (\Re, \circ)$ is still a monoid (with identity 0), usually called the *adjoint semigroup with circle operation*, or *circle semigroup* of the general ring. The circle semigroup traces back to the origins of the Jacobson radical, for a general ring $\Re$ is a Jacobson radical ring ($J(\Re) = \Re$) if and only if its circle semigroup is a group. The group of invertible elements in the monoid $(\Re, \circ)$ is exactly the set $Q(\Re) = \{q \in \Re | \exists q' \in \Re, q + q' - qq' = q + q' - q'q = 0\}$ of quasi-regular elements on the ring $\Re$. Observe also that commutation in $\Re$ is commutation in $\Re^\circ$ and that $E(\Re) = E(\Re^\circ)$. Another (equivalent) operation can also be used, the *adjoint operation* $x * y = x + y + xy$ (this is the one used primary by Jacobson [100]*). The semigroup $\Re^* = (\Re, *)$ is also monoid with $Q(\Re)$ its group of units. The map $x \mapsto -x$ is an isomorphism from $\Re^\circ$ onto $\Re^*$. For more on the adjoint and circle semigroup of a ring, notably their history, see [59]*, [90]*, [91]*, [92]*. In [151], I observed that statements involving the inverse of $1 - a$ along an element $1 - d$ (in a unital ring R) could be rewritten as statements involving the inverse of a along an element d in $(R, \circ)$. This will be put to an end throughout this section.

Another way of dealing with general rings is through unitization. Let $\Re$ be a general ring. Then a unitization of $\Re$ is a unital ring $T = \hat{\Re}$ such that $\Re$ embeds in T as a two-sided ideal. The standard unitization (sometimes called the *Dorroh extension*, even if this concept may be more general) of $\Re$ is the ring $\mathbb{Z} \oplus \Re$ with multiplication

$$(m, r)(n, s) = (mn, rs + nr + ms).$$

As observed for instance by L. Vas [206]*, the identity of $\mathbb{Z} \oplus \Re$ is $(1, 0)$ and the set of units of $\mathbb{Z} \oplus \Re$ is

$$\mathrm{U}(\mathbb{Z} \oplus \Re) = \pm (1, \mathrm{U}(\Re^*)) = \pm (1, \mathrm{U}(\Re^\circ)).$$

Finally, for a unital ring R, an interesting phenomenon occurs (described in [151]). Define a binary operation $\oplus$ by $x \oplus y = x + y - 1$. Then $R^\circ = (R, \oplus, \circ)$ is a unital ring, isomorphic to $(\mathcal{R}, +, .)$ via the map $x \mapsto 1 - x$. Though probably folklore, I found no reference in the literature. Observe that $(R^\circ)^\circ = R$ (in particular, the circle operation of $\circ$ is .).

22.1) Group regular rings

In [160], we investigated with P. Patricio various replacements of unit-regularity for elements of general rings and the general rings themselves. Four alternative concepts were discussed at the level of elements, all of which are equivalent to unit-regularity in the unital case [160, Corollary 2.5]. Group-regularity, intra group-regularity and group-domination have been defined and studied in Chapter 11. In this section, we concentrate on the concept of Q-unit-regularity. Actually, while the concepts may be distinct element-wise (see [160, Example 2.4]), they happen to coincide at the level of the (general) ring [160, Theorem 4.1] (see also Section 11.4, in particular Theorem 11.4.4).

Let $\Re$ be a general ring. By [160, Definition 2.1], $a \in \Re$ is *Q-unit-regular* (or simply unit-regular in [206]*) if $a = a^2 + aqa$ for some $q \in Q^*(\Re)$. It is group-regular if it admits an inner inverse that is group invertible.

We proved the following results regarding Q-unit-regular rings (a.k.a group-regular rings). First, a Q-unit-regular ring is regular by [160, Theorem 4.1] (since more precisely, each element has a inner inverse that is group-invertible).

Second, we can characterize Q-unit-regular rings by using unitizations. This can be done either directly or using isomorphic idempotents.

Corollary 22.1.1 ([160, Corollary 4.5])**.** Let $\Re$ be a general ring. Then the following statements are equivalent:
(1) $\Re$ is Q-unit-regular;
(2) For any unitization $\hat{\Re}$ of $\Re$, all elements of $\Re$ are unit-regular in $\hat{\Re}$;
(3) All elements of $\Re$ are unit-regular in the Dorroh extension $\mathbb{Z} \oplus \Re$ of $\Re$.

In particular, ideals of unit-regular rings are Q-unit-regular.

Recall that a regular (unital) ring is unit-regular iff its is IC (satisfies internal cancellation [110]*), iff isomorphic idempotents have isomorphic complementary idempotents ($e, f \in \mathcal{R}$ are *isomorphic* iff $e = ab$ and $f = ba$ for some $a, b \in \mathcal{R}$).

Theorem 22.1.2 ([160, Theorem 5.2])**.** Let $\Re$ be a general ring. Then the following statements are equivalent:
(1) $\Re$ is Q-unit-regular;
(2) for all $e, f \in E(\Re)$, if $e \simeq f$ then $1 - e \simeq 1 - f$ in any unitization $\hat{\Re}$ of $\Re$;
(3) for all $e, f \in E(\Re)$, if $e \simeq f$ then $1 - e \simeq 1 - f$ in the Dorroh extension $\mathbb{Z} \oplus \Re$ of $\Re$.

Another way to state this result is the following (unpublished). It does not appeal to unitization (except for its proof) but on the circle operation only. Let $e \in E(\Re)$ and $a, b \in \mathbb{Z} \oplus \Re$ such that $1 - e = (1 - a)(1 - b)$. Then a is necessarily of the form $a = (0, a')$ with $a' \in \Re$ (and then so is b), or the form $a = (2, a')$ (and so is b). In the first case, $1 - e = (1 - a')(1 - b')$ so that $e = a' \circ b'$. In the second case, $(1, -e) = (-1, -a')(-1, -b') = (1, a'+b'+a'b')$ so that $e = -a'-b'-a'b' = (-a')\circ(-b')$. From this, we deduce another equivalence.

Theorem 22.1.3 (unpublished)**.** Let $\Re$ be a general ring. Then $\Re$ is Q-unit-regular iff for all $e, f \in E(\Re)$, if $e \simeq f$ in $\Re$ then $e \simeq f$ in $\Re^\circ$.

From the above, it may be tempting to think that Q-unit-regular rings are exactly ideals of unit-regular (unital) rings, or equivalently that Q-unit-regular rings always have a unit-regular unitization. This is not the case, as we will see shortly. First, we need some terminology and results.

A unital ring R has stable range one if for all $a, b \in R$, $aR + bR = R$ implies that $(a + bc)R = R$ for some $c \in R$ (equivalently, $a + bc \in U(R)$ by [207, Theorem 2.6]*). A (general) ring $\Re$ has stable range one if for all $a \in \Re, b \in \hat{\Re}$, $(1 + a)\hat{\Re} + b\hat{\Re} = \hat{\Re}$ implies that $(1 + a + bc)\hat{\Re} = \hat{\Re}$ for some $c \in \hat{\Re}$ and some (all by [207, Theorem 3.6]*) unitization $\hat{\Re}$ of $\Re$.

As is well-known, corner rings of a unit-regular ring are unit-regular ([63]*, [82]*, [87]*, [131]*) and unital rings are unit-regular iff they are regular with stable range one (Fuchs and Kaplansky [70, Proposition 4.12]*). Regarding general rings, the following equivalences hold.

Lemma 22.1.4 ([165, Lemma 1.4]*, [32, Lemma 1]*)**.** Let $\Re$ be a regular ring. Then the following are equivalent:
(1) For each idempotent $e \in E(\Re)$ the corner ring $e\Re e$ is unit-regular.
(2) $\Re$ admits a unit-regular unitization.
(3) $\Re$ has stable range one.

In [160], we proposed the following example.

Example 22.1.1 ([160, Example 4.3])**.** Let T_0 be a regular non unit-regular unital ring. Define iteratively $T_{n+1} = \mathcal{M}_2(T_n)$ for all $n \in \mathbb{N}$, and embed each $T_n, n \in \mathbb{N}$ as the $1-1$ corner of T_{n+1}. Then define $\Re = \underset{\rightarrow}{lim} T_n$, direct limit of T_n. We claim that $\Re$ is Q-unit-regular, but has not stable range one. Indeed, we first deduce by induction that each $T_n, n \in \mathbb{N}$ is regular since matrix rings over regular rings are regular (Theorem 24 in [106]). Also $\Re$ has not stable range one since some corner rings are not unit-regular: for instance, T_0 is a non unit-regular corner ring by assumption. Let now $a \in \Re$. Then $a \in T_n$ for some $n \in \mathbb{N}$, and as T_n is regular then there exists $b \in T_n$ such that $aba = a$. Pose $B = \begin{pmatrix} b & b-1 \\ 1 & 1 \end{pmatrix} \in T_{n+1}$. Then B is group-invertible in R with group inverse $B^{\#} = \begin{pmatrix} 1 & 1-b \\ -1 & b \end{pmatrix} \in T_{n+1}$ and $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. It follows that a is group-regular, and $\Re$ is group-regular or equivalently by [160, Theorem 4.1], $\Re$ is

Q-unit-regular.

Consequently Q-unit-regular rings may not have a unit-regular unitization. Equivalently, contrary to the unital case, Q-unit-regularity does not pass to corner rings, and does not imply stable range one. On the other hand, the converse hold by [206, Proposition 1]*: a regular general ring with stable range one is Q-unit-regular.

22.2) Exchange, cleanness and special cleanness in general rings

In seminal papers, P. Ara [4]*and W.K. Nicholson [183]*used the previously defined adjoint semigroups of a general ring to extend respectively the exchange property and the cleanness property. We follow the convention of [47]*and say that an element $a \in \Re$ is *clean general* (see also [183]*) if $a = e + q^*$ for some $e \in E(\Re)$ and $q^* \in Q^*(\Re)$(or equivalently if $a = e - q^\circ$ for some $e \in E(\Re)$ and $q^\circ \in Q^\circ(\Re)$). As noted by Diesl [47, Proposition 3]*as $Q^*(\Re) \supseteq N(\Re)$ then nil-clean elements are clean general. If R is unital, then a is clean general if and only if $a+1$ is clean if and only if $-a$ is clean. A element $a \in \Re$ is *exchange general* (or simply exchange [4]*) if there exists an idempotent $e \in E(\Re)$ and $r, s \in \Re$ such that $e = ar = a \circ s$ (equivalently, $e \in a\Re \cap a \circ \Re$). As proved by P. Ara in [4]*, this property is left-right symmetric (as in the unital case).

In [153] I characterized exchange general elements by outer inverses.

Proposition 22.2.1 ([153, Proposition 3.2]). Let $a \in \Re$ general ring. Then a is exchange general iff there exists $x, z \in \Re$ such that:
1. $xax = x$;
2. $z \circ a \circ z = z$;
3. $a \circ z = ax$.

I also used outer inverses to define a second version of cleanness in general rings.

Definition 22.2.2 ([153, Definition 3.1]). Let $a \in \Re$. Then a is *g-clean* iff there exists $x, z \in \Re$ such that:
1. $xax = x$ (x is an outer inverse of a in $(\Re, .)$);
2. $z \circ a \circ z = z$ (z is an outer inverse of a in $(\Re, \circ)$);
3. $a \circ z = ax$;
4. $x + z \in Q^\circ(\Re)$.

If we also ask that $ax = xa$ and $a \circ z = z \circ a$ then we say that a is *strongly g-clean*. Another characterization of strongly g-clean elements is as follows.

Lemma 22.2.3 ([153, Lemma 3.1]). Let $a \in \Re$ be g-clean with z and g as in the definition. Then a is strongly clean iff $z \circ a = xa$.

In unital rings, clean and g-clean elements coincide.

Theorem 22.2.4 ([153, Theorem 3.1]). Let $a \in R$ unital ring. Then a is clean iff it is g-clean.

As in unital rings, we can use idempotents and principal ideals (or (e, f)-inverses).

Proposition 22.2.5 ([153, Proposition 3.1])**.** Let $a \in \mathfrak{R}$ general ring. Then a is g-clean iff there exist idempotents $e, f \in E(\mathfrak{R})$ such that:

1. $fae = ae$ and $f \circ a \circ e = a \circ e$;
2. $e \in \mathfrak{R}fae, f \in fae\mathfrak{R}$ (a has a (e, f)-inverse in $(\mathfrak{R}, .)$);
3. $e \in \mathfrak{R} \circ f \circ a \circ e, f \in f \circ a \circ e \circ \mathfrak{R}$ (a has a (e, f)-inverse in $(\mathfrak{R}, \circ)$).

We directly deduce from Proposition 22.2.1 that a g-clean element is an exchange general element with the additional property that $x + z \in Q^\circ(\mathfrak{R})$, so that g-clean are exchange general [153, Corollary 3.1]. I also proved [153, Corollary 3.2] that $a \in \mathfrak{R}$ is g-clean if and only if $-a$ is clean general.

Finally, the equivalence (1) $\iff$ (6) in Theorem 20.4.1 ([153, Theorem 4.1], [141, Lemma 2.2 and Theorem 2.4]) characterizes special clean elements as regular elements with a reflexive group invertible inverse, a characterization also valid in general rings. Thus, for $\mathfrak{R}$ a general ring, we can use this characterization as a definition of special cleanness.

Definition 22.2.6. Let $\mathfrak{R}$ be a general ring. An element $a \in \mathfrak{R}$ is g-special clean if $V(a) \cap \mathfrak{R}^\#$ is not empty.

It may not be clear that such an element is g-clean. We provide a short proof below.

Proposition 22.2.7 (unpublished)**.** In a general ring, g-special clean elements are g-clean.

Proof. Let $\mathfrak{R}$ be a general ring and $\hat{\mathfrak{R}}$ be a unitization of $\mathfrak{R}$. Let $a \in \mathfrak{R}$ be g-special clean, with reflexive group invertible inverse $z \in \mathfrak{R} \subseteq \hat{\mathfrak{R}}$. Let $e = zz^\#$ Then by [153, Theorem 6.1], $a = \bar{e} + u$ with $u \in \mathrm{U}(\hat{\mathfrak{R}})$. Morevoer, from the proof therein, u^{-1} can be written in Peirce matrix form relative to the idempotent e in the form $U^{-1} = \begin{pmatrix} z - za_2a_3z & za_2 \\ a_3z & -1 \end{pmatrix}$ (with $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_3za_2 \end{pmatrix}$). Let $q = 1 + u = a + e$ and $q' = 1 + u^{-1}$. Then $q, q' \in \mathfrak{R}$ ($q = a + e$ and $q' = (e + z(a - ae)(a - ea)z) + z(a - ae) + (a - ea)z$) and as $(-u)(-u^{-1}) = 1$ and $x \mapsto 1 - x$ is an isomorphism from $\hat{\mathfrak{R}}$ to $\hat{\mathfrak{R}}^\circ$, then $q \circ q' = (1 + u) \circ (1 + u^{-1}) = 1 - (-u)(-u^{-1}) = 1 - 1 = 0$ so that q is a unit in $\mathfrak{R}^\circ$. It follows that $-a = e - q$ is clean general, and by [153, Corollary 3.2] a is g-clean. □

The following result does not appear in my publications as well. However, it also combines different results and, as such, deserved to be present here. It relates to special cleanness of Q-unit-regular rings. As is well-known, unit-regular rings are special clean. An open question is whether this remains true for non-unital rings. I do not have an answer yet, but I provide below a proof that the Q-unit-regular ring of Example 22.1.1 is special clean. It particular, by the above results it is clean general and exchange general. However, it does not have stable range one.

Example 22.2.1 (unpublished)**.** Recall that $\mathfrak{R} = \underset{\rightarrow}{lim} T_n$ is the direct limit of T_n, where the T_n are defined iteratively with T_0 a regular non unit-regular unital ring, and

$T_{n+1} = \mathcal{M}_2(T_n)$ for all $n \in \mathbb{N}$ (and we embed each $T_n, n \in \mathbb{N}$ as the $1-1$ corner of T_{n+1}). Let $a \in \Re$. Then $a \in T_n$ for some $n \in \mathbb{N}$, and we have seen that a is unit-regular in T_{n+1}. From [161, Lemma 7.4] (or [113, Theorem 7.13]*), the matrix $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in T_{n+2} = \mathcal{M}_2(T_{n+1})$ is then special clean, and a has a reflexive inverse $z \in T_{n+2}$ that is group-invertible. This z is then group invertible in $\Re$, and a is g-special clean in $\Re$. Finally the whole ring $\Re$ is g-special clean (in particular g-clean (equiv. clean general) by Proposition 22.2.7).

22.3) Cline's formula and Jacobson's lemma in general rings

We conclude this section regarding general rings by some words on Cline's formula and Jacobson's lemma. Since Cline's formula is valid in the general setting of semigroups, it works verbatim in the case of general rings. Regarding Jacobson's lemma, there must obviously be some change in the formula compare to the unital case (also stated as Theorem 7.4.4 in part II). It involves the circle operation. For any general ring $\Re$ and any $a \in \Re$, we use the following notation

$$\Sigma_2(a) = \{a\}'' \cap \{e \in E(\Re) | e \in a\Re \cap \Re a\}.$$

By $\Sigma_2^\circ(a)$ we thus denote the set Σ_2 in the circle ring $\Re^\circ = (\Re, \oplus, \circ)$, that is

$$\Sigma_2^\circ(a) = \{a\}'' \cap \{e \in E(\Re) | e \in a \circ \Re \cap \Re \circ a\}$$

(since a and e bicommute for the circle operation iff they bicommute for the original product). By [147, Lemma 3], [151, Lemma 3.4] and Theorem 3.2.1, $e \in \Sigma_2(a)$ iff a is invertible along e and e bicommutes with a, and dually $e \in \Sigma_2^\circ(a)$ iff a is invertible along e in $\mathcal{R}^\circ$ and e bicommutes with a in $\mathcal{R}^\circ$.

Let $\Re$ be a general ring and $a, b \in \mathcal{R}$. We can state:
- Jacobson's lemma in general rings -

Theorem 22.3.1 ([151, Theorem 4.2]).
Let $e \in \Sigma_2(ab) \cap \Sigma_2^\circ(ab)$. Then $f = b(ab)^{-e}a \in \Sigma_2(ba) \cap \Sigma_2^\circ(ba)$ and

$$(ba)^{\ominus f} = \left(b\,(ab)^{\ominus e}\,a - ba\right) \circ \left(b\,(ab)^{-e}\,a\right) = b(ab)^{\ominus e}a - ba + b(ab)^{-e}a.$$

Chapter 23

Special clean elements and perspective elements - Equational characterization

23.1) Special clean elements

We present in this section some further study of special clean elements, where we focus on a very specific problem: given an element a of a ring R, is there a simple criterion to decide whether this element is special clean or not? Together with P. Patricio, we found a simple criterion based on the existence of solutions of a certain equation in a corner ring. In Chapter 24, another equation will be given, coming from a very different method.

Our method is based on the "unit-regular" characterization of special clean elements [153, Theorem 4.1], [161, Theorem 6.1], [141, Lemma 2.2]: $a \in R$ is special clean iff there exists $e \in E(R), u \in \mathrm{U}(R)$ such that $a = \bar{e} + u = au^{-1}a$.

There are at least two ways to search for such elements. A first one is to search, among the idempotents, those e such that $u = a - \bar{e}$ is invertible and $a = au^{-1}a$. A second one is to search, among the units, the inner inverses u^{-1} of a such that $a - u$ is an idempotent. It is this second method we choose to pursue, thanks to a parametrization of the set of unit-inverse due to Hartwig and Luh [86]*. This method has the defect that we must know an unit inverse v^{-1} of a to write the equation. Using chains of idempotents (Chapter 24) will pally this defect.

The principal tool is the use of Peirce decomposition and Schur complement. The Peirce decomposition of a ring R expresses R as a Morita context and conversely, any Morita context arises in this way. Precisely, given a ring R and an idempotent $e \in E(R)$ the Peirce decomposition exhibits R as the Morita context ring given by the two corner rings eRe and $\bar{e}R\bar{e}$, the bimodules $eR\bar{e}$ and $\bar{e}Re$, and multiplication as bimodule homomorphisms. The Peirce decomposition (or Peirce isomorphism) sends an element $a = \underbrace{eae}_{a_1} + \underbrace{ea\bar{e}}_{a_2} + \underbrace{\bar{e}ae}_{a_3} + \underbrace{\bar{e}a\bar{e}}_{a_4}$ to $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ (We will use upper letters for images under the isomorphism, a.k.a elements written in matrix form). The Schur complement acts for a replacement of the determinant for matrices over non-commutative rings or Morita context to prove or disprove invertibility and compute the inverse, when the coefficient in the upper left (or lower right) corner is invertible. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in a Morita context (A, B, C, D). Assume that a is invertible in the unital ring A. Then the Schur complement M/a is $d - ca^{-1}b \in D$. The matrix M is invertible iff $M/a \in \mathrm{U}(D)$, in which case the inverse is $\begin{pmatrix} a^{-1} + a^{-1}b(M/a)^{-1}ca^{-1} & -a^{-1}b(M/a)^{-1} \\ -(M/a)^{-1}ca^{-1} & (M/a)^{-1} \end{pmatrix}$.

Following this method, we obtained an equational characterization of special clean elements.

Theorem 23.1.1 ([161, Theorem 2.1])**.** Let $a \in \mathrm{ureg}(R)$ with unit inner inverse v^{-1}. Let $f = av^{-1}$ and define a function $\varphi : \bar{f}Rf \times fR\bar{f} \to \bar{f}R\bar{f}$ by

$$\varphi : (y, x) \mapsto yv_1x + yv_2 + v_3x + v_4 = (y + \bar{f})v(x + \bar{f})$$

where v has Peirce decomposition $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$ (in $fRf \oplus fR\bar{f} \oplus \bar{f}Rf \oplus \bar{f}R\bar{f}$). Then the set of special clean decompositions of a is in one-to-one correspondence with the solution set for

$$\varphi(y, x) \in U(\bar{f}R\bar{f}).$$

Precisely, any invertible u such that $a - u$ is idempotent and $au^{-1}a = a$ has Peirce decomposition

$$U^{-1} = V^{-1} \begin{pmatrix} 1 & -x \\ -y & -\varphi(y, x) + yx \end{pmatrix} = V^{-1} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varphi(y, x) \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $\varphi(y, x) \in U(\bar{f}R\bar{f})$. And conversely, any $u \in R$ of this form is invertible and satisfies that $a - u$ is idempotent and $au^{-1}a = a$.
In this case, the idempotent $\bar{e} = a - u$ has the form

$$\bar{E} = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varphi(y, x)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} V.$$

Using results of [141] we obtain another equation based on the unit v^{-1} rather than v, and the corner ring fRf. We let the Peirce decomposition of v^{-1} be $V^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ (in $fRf \oplus fR\bar{f} \oplus \bar{f}Rf \oplus \bar{f}R\bar{f}$).

Corollary 23.1.2 ([141, Corollary 4.5])**.** Let $a \in \mathrm{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Define $\psi : fR\bar{f} \times \bar{f}Rf \to fRf$ by

$$\psi : (x, y) \mapsto \mu_1 + \mu_2 y + x\mu_3 + x\mu_4 y = (x + f)v^{-1}(y + f).$$

Then the element a is special clean iff $\psi(x, y) = \mu_1 + \mu_2 y + x\mu_3 + x\mu_4 y = (x + f)v^{-1}(y + f) \in U(fRf)$ for some $x \in fR\bar{f}$, and $y \in \bar{f}Rf$.

In [161], we consider some special cases where the equations always have a solution (so that a is special clean in this cases). By $\mathrm{sr}_l(x) = 1$ we denote that x has *left stable range one*: if $sx + ty = 1$ for some s, t, y then $x + uy$ is a unit for some u (and dually for right stable range one). Then under the following conditions (and with the previous notations), a unit-regular is special clean:

(1) $\mathrm{sr}_l(v_4) = 1$ in $\bar{f}R\bar{f}$ (in particular, this holds if v_4 is unit-regular in $\bar{f}R\bar{f}$) [161, Corollary 3.3];

(2) $\mathrm{sr}_r(\mu_1) = 1$ in fRf (in particular, this holds if μ_1 is unit-regular in fRf) [161, Corollary 3.4];

(3) v_1 is unit-regular in fRf (in this case, a^2 is also special clean, see also [185, Theorem 3.14]*) [161, Corollary 3.6];

(4) $\mathrm{sr}(\bar{f}R\bar{f}) = 1$; or $\mathrm{sr}(fRf) = 1$; or $fR\bar{f} \subseteq J(R)$; $\bar{f}Rf \subseteq J(R)$ [161, Corollaries 4.1 and 4.4].

The *special cleanness* conclusion of all these premises will be further improved to *perspectivity* in the next section.

From these results, we recover by an element-wise manner some classical theorems: that a ring is unit-regular iff it is special clean [26, Theorem 1]*(since unit-regular ring have stable range one [70, Proposition 4.12]*), and that an exchange ring has stable range one iff its regular elements are unit-regular [28, Theorem 3]*iff its regular elements are special clean [33, Theorem 2.1]*.

We also deduce that a ring is uniquely special clean iff it is a strongly regular ring [161, Corollary 5.2], and that in a ring such that all skew corner rings $eR(1-e), e \in E(R)$ are contained in $J(R)$ (equivalently, idempotents are central modulo $J(R)$), then an element is regular iff it is strongly regular [161, Theorem 6.3]. This improves a result of [130]*stating that in a ring R such that $R/J(R)$ is abelian and exchange, then a is regular iff it is strongly regular, by removing the unnecessary exchange assumption. In [115, Theorem 3.13]*, it is proved that the converse also holds, and other characterizations of such rings based on chains of idempotents are provided.

By solving the equation "$\varphi(y, x)$ is a unit", we proved that any unit-regular and clean matrix $A \in \mathcal{M}_2(\mathbb{Z})$ of the form $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is special clean [161, Lemma 7.3], and that for any ring R and $a \in R$, the matrix $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(R)$ is special clean iff $a \in \mathrm{ureg}(R)$ [161, Lemma 7.4]. This last result was obtained by very different means in [113, Theorem 7.13]*.

23.2) Perspective elements

In [141], we define a new class of elements of a ring R (in link with the notion of perspectivity of direct summands), that we call perspective elements. For this section we take as a definition the following characterization [141, Theorem 3.3]: a is perspective iff it is regular and for all $f \in E(R)$ such that $Ra = Rf$ there exists a special clean decomposition $a = \bar{e} + u = au^{-1}a$ with $u \in U(R), e \in E(R)$ and $eR = fR$ (this property is actually left-right symmetric [141, Theorem 3.4]).

From the equational characterization of special clean elements in Theorem 23.1.1, we derive the corresponding result for perspective elements.

Corollary 23.2.1 ([141, Corollary 4.2]). Let $a \in \mathrm{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Define as above $\varphi : \bar{f}Rf \times fR\bar{f} \to \bar{f}R\bar{f}$ by

$$\varphi : (y, x) \mapsto yv_1x + yv_2 + v_3x + v_4 = (y + \bar{f})v(x + \bar{f}).$$

The following statements are equivalent:
(1) The element a is perspective;
(2) For all $x \in fR\bar{f}$, there exists $y \in \bar{f}Rf$ such that $\varphi(y, x) \in U(\bar{f}R\bar{f})$;
(3) For all $y \in \bar{f}Rf$, there exists $x \in fR\bar{f}$ such that $\varphi(y, x) \in U(\bar{f}R\bar{f})$.

Corollary 23.2.2 ([141, Corollary 4.4]). Let $a \in \mathrm{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Define as above $\psi : fR\bar{f} \times \bar{f}Rf \to fRf$ by

$$\psi : (x, y) \mapsto \mu_1 + \mu_2 y + x\mu_3 + x\mu_4 y = (x + f)v^{-1}(y + f).$$

Then the following statements are equivalent:
(1) a is perspective;
(2) For all $x \in fR\bar{f}$, there exists $y \in \bar{f}Rf$ such that $\psi(x, y) \in U(fRf)$;
(3) For all $y \in \bar{f}Rf$, there exists $x \in fR\bar{f}$ such that $\psi(x, y) \in U(fRf)$.

In Corollary 23.2.1, we characterize perspectivity of a unit-regular element in terms of a specific unit inner inverse. In the following corollary, we instead allow the unit inner inverse to vary, which gives more freedom choosing an x such that $\varphi(y, x)$ is a unit (in a corner ring).

Corollary 23.2.3 ([141, Corollary 4.3]). Let $a \in R$ be unit-regular. The element a is perspective iff for each decomposition of the form $a = fv$ with $f \in E(R)$ and $v \in U(R)$, there exists $y \in \bar{f}Rf$ such that $\varphi(y, 0) = (y + \bar{f})v\bar{f} \in U(\bar{f}R\bar{f})$.

Bys solving such equations, we were able to prove that regular elements squaring to 0 are perspective (they were previously known to be special clean by [185, Theorem 3.14]*and [113]*).

Proposition 23.2.4 ([141, Proposition 4.6]). Let R be a ring, and $a \in \mathrm{reg}(R)$. If $a^2 = 0$, then a is perspective.

Also, we were able to improve most of the previous results regarding special cleanness to perspectivity.

Corollary 23.2.5 ([141, Corollaries 5.1, 5.2, 5.3])**.** Let $a \in \mathrm{ureg}(R)$ with unit inner inverse v^{-1} and let $f = av^{-1}$. Under any of the following conditions, a is perspective:
(1) $\mathrm{sr}(\bar{f}R\bar{f}) = 1$;
(2) $\mathrm{sr}(fRf) = 1$;
(3) $\bar{f}RfR\bar{f} \in J(R)$.

Finally, consider a matrix $A \in \mathcal{M}_2(\mathbb{Z})$ of the form $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. We have seen that such a matrix is special clean iff it is unit-regular and clean [161, Lemma 7.3]. By contrast, it is perspective iff $a = \pm 1$ or ($a = 0$ and $b \in \{-1, 0, 1\}$) [141, Example 6.1]. For instance, the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$ is unit-regular and clean in $\mathcal{M}_2(\mathbb{Z})$ hence special clean in $\mathcal{M}_2(\mathbb{Z})$, but not perspective.

Chapter 24

Generalized inverses, chains of idempotents and $n/2$-perspectivity

Behind all the properties studied in the previous sections of the present part lie at some points questions about cancellation of modules: either the exchange property, internal cancellation, or perspectivity of direct summands. Regarding this last property, I proved that it relates to chains of associate idempotents in [139] and [141]. Independently, D. Khuruna, T.Y. Lam and P. Nielsen observed the same relationship [112]*, [115]*, [116]*. This led to a fruitful collaboration with D. Khuruna and P. Nielsen [117], [156]. In the sequel I will present the relation between chains of idempotents and other diverse concepts such as Bass's stable range one condition, quasi-continuous modules, special clean elements, strongly IC rings, (generalizations of) perspectivity or bounded generation of SL2 by elementary matrices. But before presenting these results we need as usual some definitions. In the sequel, R will be a unital ring (with in mind $R = \mathrm{End}(M)$, the endomorphism ring of a module M).

24.1) Some old definitions, and some new ones

Association chains of idempotents

Recall that two idempotents $e, f \in E(R)$ are *isomorphic* (denoted by $e \simeq f$) if eR and fR are isomorphic submodules of R_R, iff $e = ab$, $f = ba$ for some $a, b \in R$. Moreover, we can always choose such (a, b) to form a regular pair (a and b are reflexive inverses of one another). A stronger condition is that of similarity (or conjugation): e and f are similar (or conjugate) if $f = ueu^{-1}$ for some unit $u \in \mathrm{U}(R)$. Finally e and f are left (resp. right) associates, and we write $e \sim_\ell f$ (resp. $e \sim_r f$) if $Re = Rf$ or equivalently $ef = e$ and $fe = f$ (resp. $eR = fR$ or equivalently $ef = f$ and $fe = e$). Relation $\sim_\ell$ is nothing but the restriction of Green's relation $\mathcal{L}$ to the subset $E(S)$ of idempotents of the ring. Of particular importance is the following fact [112]*, [115]*, [117], [156]: $e \sim_\ell f$ iff there exists some (unique) unit $u \in 1 + (1-e)Re \subseteq \mathrm{U}(R)$ such that $f = ue$. Thus the set $(1-e)Re$ parameterizes the left associates of e. As also $eu^{-1} = e$ in this case , $f = ueu^{-1}$ is similar to e, an important fact in consideration of the invariance of relations of left and right association under similarity [156, Lemma 2.1]. Therefore, we have the following implications:

$$\text{associate} \Rightarrow \text{similar} \Rightarrow \text{isomorphic.}$$

Let $n \in \mathbb{N}$; a *left n-chain* from e to f consists of a sequence of $n+1$ idempotents $e_0; e_1; \ldots; e_n \in E(R)$ such that

$$e = e_0 \sim_\ell e_1 \sim_r e_2 \sim_\ell \cdots e_n = f.$$

We call the number n the length of the chain. Right n-chains are defined dually. When n is small, such as $n = 2$ or $n = 3$, we will write $e \sim_{\ell r} f$, respectively $e \sim_{\ell r \ell} f$, and more generally, we will write $e \sim_{(\ell r)^p} f$ (resp. $e \sim_{(\ell r)^p \ell} f$ or $e \sim_{\ell (r \ell)^p} f$) to denote that e and f are connected by a left chain of length $2p$ (resp. $2p+1$). We define *right n-chains* dually and write $e \approx f$ to denote that e and f are connected by some (left or right) association chain. Relation $\approx$ is nothing but the transitive closure of the union of $\sim_\ell$ and $\sim_r$ (and as such an equivalence relation).

Let $n \in \mathbb{N}$. Following [139], [156] and [117], and using the terminology of [116]*, we define the following properties:

(1) R is (strongly) *n-chained* if any two isomorphic idempotents are connected by both a left and a right n-chain (equivalently, by considering the complementary idempotents, any two isomorphic idempotents are connected by a left (equiv. right) association chain of length n only). In this case we also say that R satisfies $\mathcal{P}(n)$;

(2) R satisfies $\mathcal{D}(n)$ if any two conjugate idempotents are connected by a left and a right (equiv. only a left) association chain of length n;

(3) R satisfies $\mathcal{P}(n)$ (resp. $\mathcal{D}(n)$) *weakly* (or is *weakly n-chained*) if any two isomorphic (resp. conjugate) idempotents are connected by *either* a left and a right association chain of length n.

(observe that $\mathcal{P}(n)$ makes sense in any semigroup or general ring; to insist on its multiplicative form, we will sometimes apply the property to $\mathcal{M}R = (R, .)$, the monoid part of the ring R).

In [140], we use the strong and weak terminology element-wise: e and f are weakly (resp. strongly) n-chained if they are connected by either a left or a right (resp. both a left and a right) n-chain, and we note $e \sim_n^w f$ (resp. $e \sim_n^s f$) if they are weakly (resp. strongly) n-chained.

Chained and anti-chained regular elements

In [139], we use association chains to refine the notion of regularity as follows. We say that $a \in R$ is *n-chained regular* if it is regular and for all $b \in V(a)$, ab and ba are right n-chained. It is *n-anti-chained regular* if it is regular and for all $b \in V(a)$, ab and ba are left n-chained.

Extensions of perspectivity

We finally consider the notion of perspectivity and some extensions. Perspectivity is historically a lattice concept, and can be defined in any complemented lattice [14]*. Perspectivity in modules traces back J. Von Neumann [179]*and his studies on continuous geometries. It has then been reconsidered in the 60's and 70's by L. Fuchs [68]*and D. Handelman [82]*, in link with cancellation and substitution properties. The study of perspective modules and rings in full generality is more recent [69]*.

Let R be a ring, M be a (right) module and $A, A' \subseteq^{\oplus} M$ be direct summands. By $\bar{A}$ and $\bar{A}'$ we denote any two complementary summands of A and A'. We first recall the classical definitions involving perspectivity.

(1) the two direct summands A and A' are *perspective* (and we note $A \sim_{\oplus} A'$) if they share a common complementary summand ($A \oplus B = M = A' \oplus B$ for some $B \subseteq^{\oplus} M$);

(2) The module M is perspective if any two isomorphic direct summands are perspective (for any two $A, A' \subseteq^{\oplus} M$, $A \simeq A' \Rightarrow A \sim_{\oplus} A'$);

(3) The ring R is perspective if the right R-module R_R is perspective;

(4) The module M has *perspectivity transitive* if $A \sim_{\oplus} B \sim_{\oplus} C \Rightarrow A \sim_{\oplus} C$.

Following [139], we introduce some more definitions (the notion of 2-perspectivity already appears in [82]*), and let a be an endomorphism of M.

(1) The direct summands $A, A' \subseteq^{\oplus} M$ are 0-*perspective* if $A = A'$, and we also write $A \sim_{\oplus}^{0} A'$. Then, for any $p \in \mathbb{N}$, A, A' are $p+1$-*perspective* and we write $A \sim_{\oplus}^{p+1} A'$ if $A \sim_{\oplus}^{p} B \sim_{\oplus} A'$ for some $B \subseteq^{\oplus} M$ (iff A and A' are related by a sequence of $p+1$ perspectivity symbols).

(2) The module M is p-*perspective*, $p \in \mathbb{N}$ if any two isomorphic direct summands are p-perspective;

(3) The module M is $p + 1/2$-*perspective*, $p \in \mathbb{N}$ if whenever $M = A \oplus \bar{A}$ and $A \simeq A'$ ($A, A', \bar{A} \subseteq^{\oplus} M$), then $M = A' \oplus \bar{A}'$ for some $\bar{A}' \subseteq^{\oplus} M$ such that $\bar{A} \sim_{\oplus}^{p} \bar{A}'$;

(4) The endomorphism a is *kernel (resp. image) p-perspective*, $p \in \mathbb{N}$ if $\mathrm{im}(a)$, $\ker(a)$ are direct summands and $B \sim_{\oplus}^{p} \ker(a)$ for any complementary summand B of $\mathrm{im}(a)$ (resp. $B \sim_{\oplus}^{p} \mathrm{im}(a)$ for any complementary summand B of $\ker(a)$);

(5) The endomorphism a is *kernel (resp. image) $p + 1/2$-perspective* ($p \in \mathbb{N}$) if $\mathrm{im}(a)$, $\ker(a)$ are direct summands and $B \sim_{\oplus}^{p} \ker(a)$ for some complementary summand B of $\mathrm{im}(a)$ (resp. $B \sim_{\oplus}^{p} \mathrm{im}(a)$ for some complementary summand B of $\ker(a)$).

(In case $M = R_R$, the kernel of $a \in R$ is also called the right annihilator of a: $\ker(a) = r_R(a) = \{x \in R | ax = 0\}$.)

We will also write $(2p + 1)/2$-perspective instead of $p + 1/2$-perspective, so that we have a notion of $n/2$-perspective endomorphisms or modules, for any $n \in \mathbb{N}$. The ring R is $n/2$-*perspective*, $n \in \mathbb{N}$ if the right module R_R is $n/2$-perspective.

Supporting all the results of this section are the following "equivalences": direct sum decomposition correspond to idempotents, and direct summands correspond to images of regular endomorphisms. Each choice of a complementary summand corresponds to a specific reflexive inverse. All these statements are recalled precisely in [139] and [141], and can be thought as the realization of the general equivalence between add(M), the category of direct summands of finite direct sums of M, and the category of finitely generated projective modules over End(M) [57]*.

24.2) A uniform theorem

A cornerstone of the next results is the relationship between chains of different length between product of reflexive inverses. It is valid in the general setting of semigroups.

Theorem 24.2.1 ([139, Theorem 2.5])**.** Let S be a semigroup, $a \in \mathrm{reg}(S)$ and $p \in \mathbb{N}$. Then the following statement are equivalent:
(1) $ab \sim_{r\ell}^{p} \circ \sim_r ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);
(2) $ab \sim_{\ell r}^{p} ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);
(3) $ab \sim_{r\ell}^{p+1} ba$ for all $b \in V(a)$ (equiv. $b \in I(a)$) (a is $2p+2$-chained regular);
(4) $ab \sim_{r\ell}^{p+1} ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);
(5) $ab \sim_{\ell} \circ \sim_{r\ell}^{p} ba$ for some $b \in V(a)$ (equiv. $b \in I(a)$);

In particular, for any $p \geq 0$, if any $b \in V(a)$ is $2p$-chained regular then a is $2p$-anti-chained regular and the converse is true for $p \geq 1$ [139, Corollary 2.6]. In order to better understand these chained and anti-chained regular elements, we define inductively, for any semigroup S and any set $\Lambda \subseteq S$, $V^0(\Lambda) = \Lambda$ and

$$V^{p+1}(\Lambda) = V\left(V^p(\Lambda)\right) = \bigcup_{b \in V^p(\Lambda)} V(b).$$

(In case of a single element, we write $V^p(a)$ instead of $V^p(\{a\})$). By induction, the following equality also holds:

$$V^{p+1}(\Lambda) = V^p\left(V(\Lambda)\right) = \bigcup_{b \in V(\Lambda)} V^p(b).$$

We now characterize $2p+2$-chained regular elements in terms of $V^p(S^\#)$.

Proposition 24.2.2 ([139, Proposition 2.7])**.** Let S be a semigroup, $a \in \mathrm{reg}(S)$ and $p \in \mathbb{N}$. Then the following statements are equivalent:
(1) a is $2p+2$-chained regular (for all $b \in V(a)$, $ab \sim_{r\ell}^{p+1} ba$);
(2) $V^p(a) \cap S^\# \neq \emptyset$;
(3) $a \in V^p(S^\#)$.
In particular, S is $2p+2$-chained iff $reg(S) = V^p(S^\#)$.

In the particular case $p = 0$, this allows to identify 2-chained regular elements with completely regular (or strongly regular, or group invertible) elements.

We turn back to the case of rings and give two uniform theorems, one element-wise and the second one global.

Theorem 24.2.3 ([139, Theorem 3.4]). Let M be a module and $R = \mathrm{End}(M)$. Let also $a \in R$ and $n \in \mathbb{N}$.

(1) If $n = 2p$ is even then a is image (resp. kernel) p-perspective iff ab and ba are right (resp. left) $n+1$-chained for any $b \in V(a)$ iff a is $2p+1$-chained regular (resp. $2p+1$-anti-chained regular);

(2) If $n = 2p+1$ is odd then a is image (equiv. kernel) $p+1/2$-perspective iff ab and ba are right $n+1$-chained for any $b \in V(a)$ iff a is $2p+2$-chained regular;

(3) M is $n/2$-perspective iff all its regular endomorphisms are image (alternatively kernel) $n/2$-perspective.

It is crucial at this point to observe that for n odd, image $n/2$-perspectivity and kernel $n/2$-perspectivity coincide, and correspond to $n+2$-chained regularity. However, this does not relate *a priori* to anti-chained regularity. Also, for n even, image $n/2$-perspectivity and kernel $n/2$-perspectivity (equivalently, $n+1$-chained regularity and $n+1$-anti-chained regularity) are *a priori* distinct notions.

Theorem 24.2.4 ([139, Theorem 3.5]). Let M be a module, $R = \mathrm{End}(M)$, $\mathcal{M}R = (R, .)$ and $n \in \mathbb{N}$. Then the following statements are equivalent:

(1) M is $n/2$-perspective;

(2) the right module R_R (equiv. the left module ${}_RR$) is $n/2$-perspective (R is $n/2$-perspective);

(3) regular endomorphisms of M are image (equiv. kernel) $n/2$-perspective;

(4) regular elements of $\mathcal{M}R$ are $n+1$-chained regular (equiv. $n+1$-anti-chained regular);

(5) The monoid $\mathcal{M}R$ satisfies $\mathcal{P}(n+1)$.

Therefore, $n/2$-perspectivity is an "endomorphism ring property" (ER-property [129]*), in that it depend only of the endomorphism ring of the module. But even more precisely, it is a "monoid ring property", for it depends only on the monoid part of the endomorphism ring.

24.3) $n/2$-perspectivity, standard constructions and lifting hypothesis

It is known [69]*(resp. [110]*, [129]*) that a subring, a factor ring or a matrix ring over a perspective (resp. IC) ring may not be perspective, but that direct summands of perspective (resp. IC) modules are perspective (resp. IC) and corner rings of perspective (resp. IC) rings are perspective (resp. IC). Also, factoring by an ideal in the Jacobson radical preserves perspectivity (resp. IC). We consider these statements for $n/2$ perspectivity, $n \in \mathbb{N}$.

The following lemma generalize [69, Proposition 5.4 and Corollary 5.5]*(case $n = 2$, M is perspective) to smaller values of n.

Lemma 24.3.1 ([139, Lemma 5.1 and Corollary 5.2]). Let $n \leq 2$ and M (resp. R) be a $n/2$-perspective module (resp. ring). Let also N (resp. $eRe, e \in E(R)$) be a direct summand of M (resp. corner ring). Then N (resp. eRe)is $n/2$-perspective.

We now know that this property is no longer valid for 3/2-perspective rings. Indeed, from, [117, Corollary 4.4], the module $\mathbb{Z}^3$ and its endomorphism ring $\mathcal{M}_3(\mathbb{Z})$ are 3/2-perspective (or equivalently, isomorphic idempotents of $\mathcal{M}_3(\mathbb{Z})$ are 4-chained, $\mathcal{P}(4)$ holds). However, its direct summand $\mathbb{Z}^2$ and the associated corner ring $\mathcal{M}_2(\mathbb{Z})$ are known to admit chains of any length [48]*.

Proposition 24.3.2 ([139, Proposition 5.8]). Let $n \in \mathbb{N}$ and R be a $n/2$-perspective ring. Let also S a subring and J and ideal such that $R = S \oplus J$. Then S is $n/2$-perspective.

Some authors have studied lifting of associated idempotents [112]*, [162]*. Building upon their results, I proved the following facts about factor rings.

Proposition 24.3.3 ([139, Proposition 5.9]). Let R be a ring, J an ideal and $n \in \mathbb{N}$.
(1) If $n \geq 1$, $J \subseteq J(R)$ and R/J is $n/2$-perspective, then R is $n/2$-perspective.
(2) If either $J \subseteq J(R)$, or $J \subseteq \operatorname{reg}(R)$, idempotents of R/J can be lifted to R and R is $n/2$-perspective, then R/J is $n/2$-perspective.

(As shown in [139], (1) fails for $n = 0$.)

As a consequence of Proposition 24.3.3, we get that *NR rings* (a ring is NR if $\operatorname{Nil}(R)$ - set of nilpotent elements of the ring- is a subring of R) are 1/2-perspective. This applies notably to *NI rings* ($\operatorname{Nil}(R)$ is an ideal) and *UU rings* (all units are unipotent).

Proposition 24.3.4 ([139, Proposition 5.11]). Let R be a NR ring. Then R is 1/2-perspective.

Our final result considers specific constructions.

Proposition 24.3.5 ([139, Proposition 5.9]). Let $n \in \mathbb{N}$ and R be a ring.
(1) A upper triangular Morita context $\mathcal{T} = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ (where A, B are rings and ${}_AM_B$ is a bimodule) is $n/2$-perspective iff A and B are $n/2$-perspective, and $M = 0$ in case $n = 0$;
(2) If $n \geq 1$, then the ring R is $n/2$-perspective iff the ring $\mathcal{U}_k(R)$ of upper triangular matrices over R is $n/2$-perspective (for any fixed k).
(3) The ring R is $n/2$-perspective iff the power series ring $R[[X]]$ is $n/2$-perspective.
(4) The ring R is $n/2$-perspective if the polynomial ring $R[X]$ is $n/2$-perspective. The converse holds in case $n = 0$.

As $\mathbb{Z}$ is 0-perspective but $\mathcal{U}_2(\mathbb{Z})$ is not abelian, (2) fails for $n = 0$. Also, the converse of (4) fails in general [69]*.

In the next sections, I will relate n-chains (equiv. $(n-1)/2$-perspectivity) to known concepts for n small ($n = 1, 2, 3, 4$).

24.4) 1-chains and endoabelian modules

Let R be a ring and M a module. An element $a \in R$ is *right (resp. left) subcommutative* if $Ra \subseteq aR$ (resp. $aR \subseteq Ra$). A submodule A of M is *fully invariant* if for any $b \in \mathrm{End}(M)$, $bA \subseteq A$. Right (resp. left) subcommutative elements of a ring are also called *right (resp. left) duo-elements*, and a right (resp. left) subcommutative idempotent $e \in R$ is also called *left (resp. right) semicentral* (for $Re \subseteq eR \iff Re = eRe$).

Next result provides a list of equivalent characterizations of kernel 0-perspective endomorphisms. The connection with semicentral idempotents was also observed in [115, Proposition 3.8]*.

Theorem 24.4.1 ([139, Theorem 4.7]). Let M be a module and $a \in \mathrm{End}(M) = R$. The following statements are equivalent:
(1) a is kernel 0-perspective (the unique complementary summand of $\mathrm{im}(a)$ is $\ker(a)$);
(2) a is strongly regular and $\mathrm{im}(a)$ has a unique complementary summand;
(3) a is strongly regular and $aR \subseteq Rab$ for any $b \in V(a)$;
(4) a is strongly regular and $\ker(b)$ is fully invariant for any $b \in V(a)$;
(5) $\mathrm{im}(a) \oplus \ker(a) = M$ and $\ker(a)$ is fully invariant;
(6) a is strongly regular and left subcommutative;
(7) a is strongly regular and $aa^{\#}$ is right semicentral;
(8) a is regular and $b^2a = b, a^2b = a$ for any $b \in V(a)$;
(9) a is 1-anti-chained regular (regular and $ab \sim_\ell ba$ for any $b \in V(a)$);
(10) $a \in R$ is strongly regular and for all $b \in V(a)$, $ab = aa^{\#}$.

Dual characterizations of image 0-perspective endomorphisms hold. Consequently, an endomorphism is both image and kernel 0-perspective (equiv. 1-chained and 1-anti-chained) iff a is strongly regular with $aa^{\#}$ central, and we recover [25, Theorem 4.4]*, together with new characterizations.

Corollary 24.4.2 ([139, Corollary 4.8]). Let M be a module and $R = \mathrm{End}(M)$. The following statements are equivalent:
(1) M is 0-perspective (isomorphic direct summands are equal);
(2) R is abelian (M is *endoabelian*);
(3) isomorphic idempotents of $\mathcal{M}R$ are right associates (resp. left associates, resp. equal);
(4) regular elements of $\mathcal{M}R$ are completely regular and right subcommutative (resp. left subcommutative, resp. subcommutative);
(5) direct summands of M are uniquely complemented;
(6) direct summands of M are fully invariant.

We can easily add another characterization by considering conjugate idempotents or idempotents in the same $\approx$-class instead of isomorphic ones. Indeed, let R be a ring such that idempotents in the same $\approx$-class are right associates. Then they are weakly 2-chained and the ring is (strongly) 2-chained by [115, Theorem 3.13]*, so that isomorphic idempotents are 2-chained hence in the same $\approx$-class. Thus they are right associate by hypothesis hence R is abelian.

Corollary 24.4.3 (unpublished)**.** Let R be a ring. The following statements are equivalent:
(1) R is abelian ;
(2) isomorphic idempotents of $\mathcal{M}R$ are right associates;
(3) idempotents in the same $\approx$-class are right associates.

In particular, $\mathcal{D}(1) \Rightarrow \mathcal{P}(1)$.

24.5) 2-chains and strongly regular elements

We continue our study and investigate 2-chains. This relates to strong regularity. Recall that a module M is 1/2-perspective if whenever $M = A \oplus \bar{A}$ and $A \simeq A'$ then $M = A' \oplus \bar{A}'$. In [115]*, the authors describe this property in the following clever and informative form: "isomorphic direct summands share all their complements", and obtain the last equivalence of Corollary 24.5.1 ([115, Theorem 3.18]*).

Corollary 24.5.1 ([139, Corollary 4.10])**.** Let M be a module (resp. R a ring).
(1) $a \in R$ is 2-chained regular iff it is strongly regular;
(2) $a \in \mathrm{End}(M)$ is kernel 1/2-perspective iff it is image 1/2-perspective iff $im(a) \oplus \ker(a) = M$;
(3) $\mathcal{M}R$ is 2-chained iff $\mathrm{reg}(\mathcal{M}R) = \mathcal{M}R^{\#}$;
(4) M (resp. R) is 1/2-perspective iff regular endomorphisms are strongly regular (resp. $\mathrm{reg}(R) = R^{\#}$).

In [139], I also discussed the semigroup case, notably when the semigroup is π-regular. To the best of my knowledge, very few results are known about rings whose regular elements are all strongly regular in full generality, and none involved module arguments until the very recent work of Khuruna and Nielsen [115, Theorem 3.18]*. A fine characterization involving square stable range one is given in [114, Theorem 5.4]*. An element a of a ring R is said to have (right) square stable range one (ssr(1)) if $aR + bR = R$ implies that $a^2 + bx$ is a unit for some $x \in R$. A ring R has square stable range on if all its elements have. The authors prove that for a ring R, having all they regular elements strongly regular or with square stable range one are equivalent properties, and call such rings strongly IC (for they are always IC). Thus we deduce from Corollary 24.5.1 and [114, Theorem 5.4]*that R is 1/2-perspective iff R is strongly IC ($\mathrm{reg}(R) = R^{\#}$) iff regular elements of R have ssr(1). Another characterization [115, Theorem 3.13]*will be discussed shortly.

Second, we prove that in a ring, 2-anti-chained regular elements are necessarily 2-chained regular, and characterize them by means of the Jacobson radical.

Proposition 24.5.2 ([139, Proposition 4.16])**.** Let $a \in R$. The following statements are equivalent:
(1) a 2-anti-chained regular;
(2) a is regular and, for all $b \in V(a)$, $ab - ba \in J(R)$;
(3) a is 2-chained regular (strongly regular) and 2-anti-chained regular.

By [139, Example 4.5], there exist 2-chained regular elements that are not 2-anti-chained regular. This element-wise result has the following global consequence.

Corollary 24.5.3 ([139, Corollary 4.17])**.** R is 1/2-perspective iff isomorphic idempotents of R are equal modulo the Jacobson radical.

It happens that, concomitantly and independently to the redaction of [139], D. Khurana and P.P. Nielsen proved an even more precise result.

Theorem 24.5.4 ([115, Theorem 3.13]*)**.** For a ring R, the following are equivalent:
(1) Any two isomorphic idempotents are strongly 2-chained;
(2) $\operatorname{reg}(R) = \operatorname{sreg}(R)$;
(3) $\operatorname{ureg}(R) = \operatorname{sreg}(R)$;
(4) $\operatorname{sp.cl}(R) = \operatorname{sreg}(R)$;
(5) Any two idempotents in the same $\approx$-class are weakly 2-chained;
(6) Idempotents of R are central modulo the Jacobson radical.

In particular, $\mathcal{D}(2)$ (weakly) $\Rightarrow \mathcal{P}(2)$. In [140] (see Section 24.8.2), we add another characterization based on transitivity of weak/strong 2-chaining.

24.6) Perspective rings, 3-chains and perspective elements

Perspective modules and rings have been studied thoroughly in [69]*. One of their main result is that a ring R has stable range one iff the ring $\mathcal{M}_2(R)$ is perspective. In [156, Proposition 4.1], we provide a very short proof of this result using association chains. In [141], we propose an element-wise study of such rings, via the introduction of *perspective elements*. Right (resp. left) perspective elements of a ring are shown to correspond to 3-chained-regular and 3-anti-chained-regular elements respectively, and right (resp. left) perspective elements an endomorphism ring are shown to correspond to image (resp. kernel) 1-perspective endomorphisms. But more importantly we prove that the notion is left-right symmetric [141, Theorem 3.4], making great use of [117, Lemma 3.7].

We recall the definition of perspective elements and its various characterizations below, making use of the left-right symmetry of the notion.

Definition 24.6.1 ([141, Definition 3.2])**.** Let R be a ring, and $a \in R$. We say that a is *perspective* if it is regular and any complementary summand of $r_R(a)$ is perspective with aR.

The set of perspective elements of R will be denoted by per(R).

The next theorem characterizes perspective elements in terms of clean and special clean decompositions, reflexive inverses, idempotents and direct summands. All prime statements follow from the left-right symmetry of the notion [141, Theorem 3.4].

Theorem 24.6.2 ([141, Theorem 3.3] and [141, Theorem 3.4])**.** Let R be a ring, and $a \in R$. The following statements are equivalent:

(1) a is regular and any complementary summand of $r_R(a)$ is perspective with aR (a is perspective);

(1′) a is regular and any complementary summand of $l_R(a)$ is perspective with Ra;

(2) (aR, bR characterization) a is regular and for all $b \in V(a)$, aR and bR are perspective (as right R-submodules of R_R);

(2′) (Ra, Rb characterization) a is regular and for all $b \in V(a)$, Ra and Rb are perspective (as left R-submodules of ${}_RR$);

(3) (Clean characterization) a is regular and for all $f \in E(R)$ such that $Ra = Rf$ there exists a clean decomposition $a = \bar{e} + u$ with $u \in U(R), e \in E(R)$ and $eR = fR$;

(3′) (Clean characterization) a is regular and for all $f \in E(R)$ such that $aR = fR$ there exists a clean decomposition $a = \bar{e} + u$ with $u \in U(R), e \in E(R)$ and $Re = Rf$;

(4) (Special clean characterization) a is regular and for all $f \in E(R)$ such that $Ra = Rf$ there exists a special clean decomposition $a = \bar{e}+u = au^{-1}a$ with $u \in U(R), e \in E(R)$ and $eR = fR$;

(4′) (Special clean characterization) a is regular and for all $f \in E(R)$ such that $aR = fR$ there exists a special clean decomposition $a = \bar{e}+u = au^{-1}a$ with $u \in U(R), e \in E(R)$ and $Re = Rf$;

(5) (Group inverse characterization) a is regular and for all $b \in V(a)$, there exists $z \in V(a) \cap R^{\#}$ such that $zR = bR$;

(5′) (Group inverse characterization) a is regular and for all $b \in V(a)$, there exists $z \in V(a) \cap R^{\#}$ such that $Rz = Rb$;

(6) (Idempotent characterization) a is regular and for all $b \in V(a)$, $ab \sim_{r\ell r} ba$ (a is 3-chained-regular);

(6′) (Idempotent characterization) a is regular and for all $b \in V(a)$, $ab \sim_{\ell r\ell} ba$ (a is 3-anti-chained-regular);

(7) ($l_R(a), l_R(b)$ characterization) a is regular and for all $b \in V(a)$, $l_R(a)$ and $l_R(b)$ are perspective (as left R-submodules of ${}_RR$);

(7′) ($r_R(a), r_R(b)$ characterization) a is regular and for all $b \in V(a)$, $r_R(a)$ and $r_R(b)$ are perspective (as right R-submodules of R_R);

(8) (Dual characterization) a is regular and any complementary summand of Ra is perspective with $l_R(a)$;

(8′) (Dual characterization) a is regular and any complementary summand of aR is perspective with $r_R(a)$.

From [69, Theorem 4.2, point (4)]*, [141, Theorem 3.4], [139, Proposition 4.19] and [117, Corollary 3.11] we obtain the following characterization of perspective modules and rings.

Proposition 24.6.3 ([69, Theorem 4.2]*, [139, Proposition 4.19], [141, Theorem 3.4] and [117, Corollary 3.11])**.** Let M be a module, $R = \mathrm{End}(M =$ and $\mathcal{M}R = (R, .)$. Then M is perspective iff R_R is perspective (R is perspective) iff regular elements of R are perspective iff the monoid $\mathcal{M}R$ satisfies $\mathcal{P}(3)$ iff it satisfies $\mathcal{P}(3)$ weakly.

We now give some further results regarding perspective elements. First, strongly regular elements (in particular, idempotents and units) are perspective in any ring [141, Lemma 3.6]. I also proved [141, Corollary 3.10] that perspective elements are uniquely special clean iff $V(a)$ is a singleton iff a is strongly regular and $aa^{\#}$ is central. And finally, I relate perspective elements to a certain "stable range" property. Indeed, it is known [110, Theorem 3.5]*that a regular element of a ring has left (equiv. right) stable range one iff it is unit-regular. In rings with stable range one (in particular unit-regular rings) regular elements have right and left *idempotent stable range one*, where $a \in R$ has right idempotent stable range one if for all $b \in R$, $ax + by \in \mathrm{U}(R)$ for some $x, y \in R$ implies that $a + be \in \mathrm{U}(R)$ for some $e \in E(R)$. We prove that perspective elements are precisely regular elements with *outer inverse right stable range 1*, where $a \in R$ has outer inverse right stable range one if $aR + bR = R$ for some $b \in R$ implies that $a + bx \in U(R)$ for some outer inverse $x \in R$ of b. By left right-symmetry of perspective elements, outer inverse stable range one is a left-right symmetric notion for regular elements, that is the following statements are equivalent for any $a \in \mathrm{reg}(R)$:

(1) If $aR + bR = R$ for some $b \in R$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b;

(2) If $Ra + Rb = R$ for some $b \in R$ then $a + xb \in U(R)$ for some outer inverse $x \in R$ of b.

Proposition 24.6.4 ([141, Proposition 5.5])**.** Let R be a ring and $a \in \mathrm{reg}(R)$. Then the following statements are equivalent:
(1) a is perspective;
(2) If $aR + bR = R$ for some $b \in R$ then a admits a special clean decomposition $a = \bar{e} + u = au^{-1}a$ for some $e \in E(R), u \in U(R)$ such that $\bar{e}R \subseteq bR$;
(3) If $aR + bR = R$ for some $b \in R$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b ($xbx = x$) such that $aR \cap bxR = 0$;
(4) If $aR + bR = R$ for some $b \in R$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b (a has outer inverse right stable range one);
(5) If $aR + bR = R$ for some $b \in \mathrm{reg}(R)$ then $a + bx \in U(R)$ for some outer inverse $x \in R$ of b;
(6) If $aR + \bar{f}R = R$ for some $f \in E(R)$ then a admits a clean decomposition $a = \bar{e} + u$ for some $e \in E(R), u \in U(R)$ such that $\bar{e}R \subseteq \bar{f}R$.
Dual statements hold.

24.7) 4-chains, 3/2-perspective modules and special clean elements

Recall that by definition, a module M is 3/2-perspective if for any two isomorphic direct summand $A, A' \subseteq^{\oplus} M$, any complementary summand of A is perspective to some complementary summand of A'. Collecting all our knowledge regarding the various characterizations of special clean elements, we obtain Proposition 24.7.1. The equivalences $\operatorname{reg}(R) = \operatorname{sp.cl}(R) \iff \mathcal{M}R$ satisfies $\mathcal{P}(4)$ and (1) $\iff$ (4) therein were also obtained independently by D. Khuruna and P.P. Nielsen [115, Proposition 3.19 and Theorem 4.1]*, where kernel 3/2-perspective endomorphisms are termed *pc-regular* (and they say that $\operatorname{im}(a)$ and $\ker(a)$ are perspective in complement).

Proposition 24.7.1 ([139, Theorem 2.5], [139, Proposition 4.20])**.** Let M be a module, and $a \in R = \operatorname{End}(M)$. Let also $\mathcal{M}R = (R, .)$. Then the following statements are equivalent:
(1) a is image (equiv. kernel) 3/2-perspective;
(2) a is 4-chained regular;
(3) aR and bR are perspective, for some $b \in V(a)$;
(4) a has a completely regular reflexive inverse (as an element of $\mathcal{M}R$);
(5) a is special clean (as an element of the ring R).
Also M is 3/2-perspective iff R_R is 3/2-perspective iff $\operatorname{reg}(R) = V(R^{\#}) = \operatorname{sp.cl}(R)$ iff $\mathcal{M}R$ satisfies $\mathcal{P}(4)$.

In connection with this proposition, let me mention some close results where the authors consider conjugate rather than isomorphic idempotents, or equivalently unit-regular rather than regular elements. By [115, Corollary 4.5]*, a right self-injective ring R satisfies $\mathcal{D}(4)$ (conjugate idempotents are connected by a left and right association chain of length 4), or equivalently $\operatorname{ureg}(R) = \operatorname{sp.cl}(R)$. And by [115, Theorem 4.11]*, for any quasi-continuous M, its endomorphism ring $R = \operatorname{End}(M)$ satisfies $\mathcal{D}(4)$, or equivalently $\operatorname{ureg}(R) = \operatorname{sp.cl}(R)$.

24.8) More equations for chains, and consequences

The following criterion characterizes when isomorphic idempotents are association chained. It has a nice interpretation in the corresponding Peirce decomposition.

Theorem 24.8.1 ([156, Theorem 2.5] and [117, Theorem 3.1])**.** Let R be a ring, $n \in \mathbb{N}$ and let $a, b \in R$ be a pair of reflexive inverses. Setting $e = ab$ and $f = ba$, then there is a left $n+2$-chain from f to e iff there exist $z_1, z_2, \ldots, z_n$ with

$$z_i \in \begin{cases} (1-e)Re \text{ if } i \text{ is odd} \\ eR(1-e) \text{ if } i \text{ is even} \end{cases}$$

such that $ea(1+z_n)(1+z_{n-1})\ldots(1+z_2)(1+z_1)e \in \mathrm{U}(eRe)$.

Consider the cases $n = 0, 1, 2$. With the above notations, a and b can be written in Peirce matrix form $A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 0 \\ b_3 & 0 \end{pmatrix}$ with $a_1b_1 + a_2b_3 = 1_{eRe}$.

$\underline{n=0}$ The theorem is just Theorem 1.1.2: $ba \sim_{\ell r} ab$ iff a_1 is a unit in eRe. In this case, this is also equivalent with $faf \in \mathrm{U}(fRf)$ [156, Theorem 2.5]. Equivalently, $ab \sim_{\ell r} ba$ iff b_1 is a unit in eRe;

$\underline{n=1}$ $ba \sim_{\ell r\ell} ab$ iff there exists $z_1 \in (1-e)Re$ such that $a_1 + a_2 z_1$ is a unit in eRe;

$\underline{n=2}$ $ba \sim_{\ell r\ell r} ab$ iff there exists $z_1 \in (1-e)Re$ and $z_2 \in eR(1-e)$ such that $a_1 + (a_1 z_2 + a_2)z_1$ is a unit in eRe.

Theorem 24.8.1 has many interesting consequences. For instance, in [115]*, the authors propose another characterization of 2-chained idempotents. Let $e, f \in R$ and consider Peirce decompositions relative to e. Then $e \sim_{\ell r} f$ iff

$$f = \begin{pmatrix} 1-sr & s \\ r-rsr & rs \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r & 0 \end{pmatrix} \begin{pmatrix} 1-sr & s \\ 0 & 0 \end{pmatrix}$$

for some $r \in (1-e)Re$ and $s \in eR(1-e)$. Letting $a = \begin{pmatrix} 1-sr & s \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ r & 0 \end{pmatrix}$, we observe that $ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e$ and that a, b are reflexive inverses. First, we recover that $e = ab \sim_{\ell r} ba = f$ since $b_1 = e \in U(eRe)$. Second, we deduce from Theorem 24.8.1 that e and f are also right 2-chained (hence strongly 2-chained) iff $eae = e - sr \in U(eRe)$. This will be critical to study transitivity of chaining (Section 24.8.2, based on [140]).

By Theorem 24.2.1 ([139, Theorem 2.5]) the idempotents ba and ab are left 4-chained iff a is 4-chained-regular iff a is special clean. This gives a very different equation than the ones obtained in [161, Theorem 2.1] (Theorem 23.1.1) and [141, Corollary 4.5] (Corollary 23.1.2). The main advantage is that we no longer require the knowledge of a **unit inner inverse** of a (an **arbitrary inner inverse** b is now sufficient). The same reasoning also leads to a new equational characterization of perspective elements.

Corollary 24.8.2 (unpublished)**.** Let R be a ring, and $a \in \mathrm{reg}(R)$ be a regular element with inner inverse b. Let $e = ab, a_1 = eae$ and $a_2 = ea(1-e)$. Then:
(1) a is perspective iff for all $z_2 \in eR(1-e)$, $a_1 + (a_1 z_2 + a_2)z_1$ is a unit in eRe for some $z_1 \in (1-e)Re$;
(2) a is special clean iff $a_1 + (a_1 z_2 + a_2)z_1$ is a unit in eRe for some $z_1 \in (1-e)Re$ and $z_2 \in eR(1-e)$.

To compare properly with Theorem 23.1.1, assume that $b = v^{-1}$ is a unit inner inverse. Then $a_1 = eae = a^2 b = a^2 v^{-1} = eve = v_1$ and $a_2 = ea(1-e) = ea - eae = a - eae = ev - eve = v_2$. The criterion for special cleanness of a of Theorem 23.1.1 then also reads $ya_1x + ya_2 + v_3x + v_4 \in \mathrm{U}(\bar{e}R\bar{e})$.

To illustrate Corollary 24.8.2 and also introduce the next results, consider the following problem. In [141, Example 6.2], I proved special cleanness of the matrix $A = \begin{pmatrix} 1+X & X^2 \\ 0 & 0 \end{pmatrix}$ on the ring $\mathcal{M}_2(F[X])$ (where F is a field) by using Theorem 23.1.1. Hereafter we use Corollary 24.8.2. Let $B = \begin{pmatrix} 1-X & 0 \\ 1 & 0 \end{pmatrix}$. Then $B \in I(A)$

and $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $a_1 = 1 + X$ and $a_2 = X^2$. We compute the Euclidean algorithm: $a_2 = X^2 = Xa_1 - X$ and $a_1 = 1 + X = (-1)(-X) + 1$. Thus $1 = a_1 + 1(-X) = a_1 + 1(-Xa_1 + a_2)$. Thus $a_1 + (a_1 z_2 + a_2) z_1 = 1 \in \mathrm{U}(F(X))$ for $z_1 = 1, z_2 = -1 \in F(X)$. It follows from Corollary 24.8.2 that A is special clean.

More generally, if i is odd (resp. even), then in Peirce matrix form $1 + z_i = \begin{pmatrix} 1 & 0 \\ z_i & 0 \end{pmatrix}$ (resp. $1 + z_i = \begin{pmatrix} 1 & z_i \\ 1 & 0 \end{pmatrix}$. This also allows to identify association chains with certain (standard) division chains, and thus to relate some of our results to number theory. The non-commutative case needs some additional work, but in the commutative case division chains arise as in the classical Euclidean algorithm illustrated above. The general theory is presented in [117].

Recall that given a ring S, a pair $(a, b) \in S^2$ is *right unimodular* if $aS + bS = S$. Left unimodular pairs are defined dually, and the definition extends to $n-$uples in a straightforward manner.

Theorem 24.8.3 ([117, Theorem 3.2]). Let $n \in \mathbb{N}$, and let S be a Dedekind-finite ring such that every nontrivial idempotent in $R = M_2(S)$ is isomorphic to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Every left and every right unimodular pair from S has a division chain of ordered termination length at most $n + 1$ iff R satisfies $\mathcal{P}(n + 2)$.

Corollary 24.8.4 ([117, Corollary 3.3]). Let K be a number field, let X be a finite set of valuations on K including the archimedean valuations, and let

$$\mathcal{O}_X = \{x \in K | x = 0 \text{ or } \nu(x) \geq 0 \text{ for all } \nu \notin X\}$$

be the ring of X-integers in K. If $\mathcal{O}_X$ has infinitely many units, then the ring $R = M_2(\mathcal{O}_X)$ satisfies $\mathcal{P}(9)$, and under a generalized Riemann hypothesis (GRH) it satisfies $\mathcal{P}(6)$.

Corollary 24.8.5 ([117, Corollary 3.5]). Let S be a nontrivial localization of $\mathbb{Z}$. Then $R = M_2(S)$ satisfies $\mathcal{P}(5)$, and it satisfies $\mathcal{P}(4)$ under GRH.

24.8.1) From weakly-chained rings to strongly-chained rings

Theorem 24.8.1 has another very interesting consequence. It is a key ingredient of next lemma, which in turn allows to move from weakly 3-chained rings to strongly 3-chained rings and to prove the left-right symmetry of the notion of perspective elements.

Lemma 24.8.6 ([117, Lemma 3.7]). Let $a, b \in R$ be reflexive inverses, and put $e = ab, f = ba \in E(R)$. There exists an element $a' \in eR$ such that $a'b = e$ with the property that if $e \sim_{\ell r \ell} g = ba'$ then $e \sim_{r \ell r} f$.

Theorem 24.8.7 ([117, Theorem 3.10])**.** Let R be a ring, let $e \in E(R)$, and let X be the $\approx$-equivalence class of e. If every elements of X is connected to e either by a left or a right 3-chain, then every such element is so connected by both a left and right 3-chain.

Corollary 24.8.8 ([117, Corollary 3.11])**.** A ring R is perspective if and only if any two isomorphic idempotents are connected by either a left or a right 3-chain.

Theorem 24.8.9 ([141, Theorem 3.4])**.** Let R be a ring, and $a \in \mathrm{reg}(R)$. Then any complementary summand of $r_R(a)$ is perspective with aR (a is perspective) iff any complementary summand of $l_R(a)$ is perspective with Ra (equiv. perspectivity of elements is a left-right symmetric notion).

24.8.2) Transitivity of chaining

In any ring R, perspectivity is transitive (for any three direct summands $A, B, C \subseteq^{\oplus} R_R$, $A \sim_{\oplus} B \sim_{\oplus} C \Rightarrow A \sim_{\oplus} C$) iff relation $\sim_{r\ell r}$ is transitive on $\mathrm{idem}(R)$ from [48, Lemma 6.3]*. And, still by [48, section 6]*, this holds iff any left 3-chain is a right 3-chain. In general, transitivity of perspectivity does not imply perspectivity. For instance, while unit-regular rings are known to be perspective (and in particular have perspectivity transitive), there are examples due to Bergman of regular rings that have perspectivity transitive but are not perspective.

But what about smaller chains?

(1) Is there a relationship between strongly (resp. weakly) 1-chained rings and rings where strong (resp. weak) 1-chaining is transitive?

(2) Is there a relationship between strongly 2-chained rings and rings where weak (or strong) 2-chaining is transitive?

In [140], I proved the following results. The proofs are based on the different equational characterizations of chained idempotents of Section 24.8.

Theorem 24.8.10 ([140, Theorem 2.4])**.** Let R be a ring. Then the following statements are equivalent.
(1) Idempotents of R are either right semicentral or left semicentral;
(2) Isomorphic idempotents (equiv. in the same $\approx$-class) are weakly 1-chained;
(3) Weak 1-chaining is transitive in R (equiv. in each $\approx$-class).

Theorem 24.8.11 ([140, Theorems 2.1 and 2.2])**.** Let R be a ring such that strong or weak 2-chaining is transitive. Then idempotents are central modulo the Jacobson radical.

Using [115, Theorem 3.13]*, we obtain the following corollary.

Corollary 24.8.12 ([140, Corollary 2.3])**.** Let R be a ring. Then the following statements are equivalent.
(1) Idempotents of R are central modulo $J(R)$;
(2) Isomorphic idempotents of R are strongly 2-chained;
(3) Idempotents of R in the same $\approx$-class are weakly 2-chained;
(4) Strong 2-chaining is transitive in R (equiv. in each $\approx$-class);
(5) Weak 2-chaining is transitive in R (equiv. in each $\approx$-class).

24.8.3) Transitivity of perspectivity and IC

In [156], we consider the following question. If an IC ring has perspectivity transitive, is it perspective? This question is motivated as follows. For a **regular** ring R, the following conditions are well known to be equivalent:

(1) The ring R is unit-regular.

(2) The ring R is IC.

(3) The ring R is perspective ring

(4) The ring $\mathcal{M}_2(R)$ has transitivity of perspectivity.

(5) The ring R has stable range one.

They are not equivalent in the non-regular case, but still some implications hold:

- $\mathcal{M}_2(R)$ has transitive perspectivity iff R has stable range one [116, Theorem 2.5]*, in which case R is perspective;
- If R is perspective then R has transitive perspectivity and R is an IC ring.

Also, IC rings are very close to unit-regular rings; actually, a ring R is IC iff $\text{reg}(R) = \text{ureg}(R)$. This raises the tantalizing possibility that any IC ring with transitive perspectivity must be perspective. In [156], we construct a counterexample, thus proving the following result.

Theorem 24.8.13 ([156, Theorem 1.1])**.** There exists an IC ring with transitive perspectivity that is not a perspective ring.

Now, I present this counterexample.

Let D be the subset of $\mathbb{Z} - \{0\}$ consisting of those integers whose prime factors are all congruent to $\pm 1 \pmod 8$. Note that D is a multiplicatively closed subset of $\mathbb{Z}$, and fix $T = D^{-1}\mathbb{Z}$, which is a subring of $\mathbb{Q}$. It makes sense to talk about congruence modulo 8 in T; also note that any element of T that is not congruent to $\pm 1 \pmod 8$ is not a unit.

Fix $R = \begin{pmatrix} T & 4T \\ 4T & T \end{pmatrix}$, which is a subring of $S = \mathcal{M}_2(T)$. In [156], we prove that

(1) R is an IC ring;

(2) perspectivity is transitive;

(3) R is not a perspective ring.

From Corollary 24.8.5 ([117, Corollary 3.5]) we also know that $S = \mathcal{M}_2(T)$ satisfies $\mathcal{P}(5)$, and $\mathcal{P}(4)$ under GRH. A simple application of Theorem 24.8.1 and use of Dirichlet's theorem on primes in arithmetic progressions (as in [156], but twice) prove that this is indeed the case. And the same matrices as in [156] prove that S does not satisfies $\mathcal{P}(3)$. Also, the right S module S_S is not quasi-continuous.

24.8.4) Bounded generation of $SL_2(S)$, and length of associations chains

The generation of $SL_2(S)$ by (products of) elementary matrices is a long-standing question in ring theory, as is the search of universal bounds for the size of the products.

It is known that any universal bound on the lengths of division chains, for unimodular pairs over a commutative ring S, gives a bound on the number of elementary matrices needed to generate $SL_2(S)$. Indeed, by [102, Theorem 3.6]*the latter bound is at most 4 greater. On the other hand, we have seen that for S a projective-free ring, the division chains have ordered termination length at most $n+1$ iff R satisfies $\mathcal{P}(n+2)$. Thus for a commutative ring S a universal bound on association chains gives a universal bound on the number of elementary matrices needed to generate $SL2(S)$ (at most 3 greater).

It is also well known that if S has n in its stable range (every (right) unimodular row of size $n+1$ is reducible), and $m \gg n$, then $SL_m(S)$ is generated by a bounded number of elementary matrices.

Next theorem proves that if S has $n \geq 2$ in its stable range, most matrix rings over S satisfy $\mathcal{P}(4)$.

Theorem 24.8.14 ([117, Theorem 4.3])**.** Let $n \geq 2$ be an integer, and assume that S is a projective-free ring with n in its stable range. If $m \geq 4n-5$, then $R = M_m(S)$ satisfies $\mathcal{P}(4)$. If S does not have 1 in its stable range, then $\mathcal{P}(3)$ fails (i.e., R is not a perspective ring).

As $\mathbb{Z}$ has stable range 2, then all regular elements in $\mathcal{M}_m(\mathbb{Z})$ are special clean for any $m \geq 3$ [117, Corollary 4.4], while $\mathcal{M}_m(\mathbb{Z})$ is never perspective. On the other hand, there is no finite bound on association chains in $\mathcal{M}_2(\mathbb{Z})$ [48]*.

Conclusion, open problems and future work

In this chapter, we have seen that, as is the case for semigroups, generalized inverses and idempotents are of great use to study ring elements and special classes of rings (even properties involving the sum operation). And that in many cases, this allows to work with non-unital rings. On the other hand, thanks to some new tool brought by unital ring theory (Peirce decomposition, use of modules, creation of units,...), we are able to push further the study and prove new results.

While some questions were solved in this chapter, many new questions emerged. Below are some lines of future research:

(1) the characterization of clean elements by Bott-Duffin inverses, and the study of Jacobson's lemma for outer inverses suggest a further study of a new class of elements: those $a \in R$ such that a is invertible along e and $1-a$ is invertible along $1-e$ for some idempotent e. We have seen that they form a proper subset of the clean elements, that contains the strongly clean elements and the strongly regular elements. Second, for a given $a \in R$ in this class, among the possible idempotents bicommuting with a, is there a maximal one M? The inverse a^{-M} would then be a *binatural inverse* of a, whose properties are worth to study. The class of such binaturally invertible a is quite large: any group or (generalized) Drazin invertible elements is binaturally invertible;

(2) we have seen that group-regular general rings act as a good replacement for unit-regular (unital) rings. And as is well-known unit-regular rings have stable range one, hence are perspective and special clean. Also, we have seen that all these three properties having non-unital analogs (the last two by using chains of idempotents). But group-regular general rings may not have stable range one. Thus it is an open question whether group-regular general rings are 3-chained (a replacement for perspectivity), or 4-chained (equiv. all elements have a group-invertible reflexive inverse, a replacement for special cleanness). These questions may be very challenging, for the following two reasons: first, we cannot use consider a general ring $\mathcal{R}$ as the endomorphism ring of some module, and therefore cannot use perspectivity of submodules. Second, our studies of chains of idempotents make great use of complementary idempotents (in particular through Peirce decompositions);

(3) some questions are also still open in the unital case, most of them considering symmetry properties. For instance: are weakly 4-chained rings strongly 4-chained? Or equivalently, are 4-anti-chained-regular elements 4-chained-regular? Are there sum decompositions for 5-chained or 5-anti-chained regular elements? Do they coincide?

(4) finally, another intriguing question is the following: are there n-chained rings not $n-1$-chained, for any $n \geq 1$? And if so, can we find a universal construction?

Bibliography

[1] E. Akalan and L. Vaš. Classes of almost clean rings. *Algebr. Representat. Theory*, 16(3):843–857, 2013.

[2] A.Nagy. Special classes of semigroups. In *Adv. Math. (Dordrecht)*, volume 1. Kluwer Academic Publishers, Dordrecht, 2001.

[3] P.N. Ánh and L. Márki. Morita equivalence for rings without identity. *Tsukuba J. Math.*, 11(1):1–16, 1987.

[4] P. Ara. Extensions of exchange rings. *J. Algebra*, 197(2):409–423, 1997.

[5] K. Auinger, T. E. Hall, N. R. Reilly, and S. Zhang. Congruences on the lattice of pseudovarieties of finite semigroups. *Int. J. Algebra Comput.*, 7(04):433–455, 1997.

[6] G. Azumaya. Strongly π-regular rings. *J. Fac. Sci. Hokkaido Univ.*, 13:34–39, 1954.

[7] J.K. Baksalary. A relationship between the star and minus orderings. *Linear Algebra Appl.*, 82:163–167, 1986.

[8] J.K. Baksalary, F. Pukelsheim, and G.P.H. Styan. Some properties of matrix partial orderings. *Linear Algebra Appl.*, 119:57–85, 1989.

[9] R. Bardenet, F. Lavancier, X. Mary, and A. Vasseur. On a few statistical applications of determinantal point processes. *ESAIM: Proceedings and Surveys*, 60:180–202, 2017.

[10] Adi Ben-Israel and Thomas NE Greville. *Generalized inverses: theory and applications*, volume 15. Springer-Verlag, new york, 2003.

[11] J. Benítez and E. Boasso. The inverse along an element in rings. *Electron. J. Linear Algebra*, 31(1):572–592, 2016.

[12] J. Benítez and E. Boasso. The inverse along an element in rings with an involution, Banach algebras and C^*-algebras. *Linear Multilinear Algebra*, 65(2):284–299, 2017.

[13] J. Benítez, E. Boasso, and H. Jin. On one-sided (b, c)-inverses of arbitrary matrices. *Electron. J. Linear Algebra*, 32:391–422, 2017.

[14] Garrett Birkhoff. *Lattice theory*, volume 25. American Mathematical Soc., 1940.

[15] T.S. Blyth, E. Giraldes, and M.P.O. Marques-Smith. Associate subgroups of orthodox semigroups. *Glasg. Math. J.*, 36(2):163–171, 1994.

[16] T.S. Blyth and J.B. Hickey. Rp-dominated regular semigroups. *Proc. R. Soc. Edinb. A: Math.*, 99(1-2):185–191, 1984.

[17] T.S. Blyth and R. McFadden. Unit orthodox semigroups. *Glasg. Math. J.*, 24(1):39–42, 1983.

[18] E. Boasso. Further results on the (b, c)-inverse, the outer inverse $A^{(2)}_{T,S}$ and the Moore–Penrose inverse in the Banach context. *Linear Multilinear Algebra*, 67(5):1006–1030, 2019.

[19] E. Boasso and G. Kantún-Montiel. The (b, c)-inverse in Rings and in the Banach Context. *Mediterr. J. Math.*, 14(3):112, 2017.

[20] S. Bogdanović, M. Cirić, and T. Petković. Uniformly π-regular rings and semigroups: a survey. *Zbornik Radova*, (17):5–82, 2000.

[21] M. Brion and A. Rittatore. The structure of normal algebraic monoids. *Semigroup Forum*, 74(3):410–422, 2007.

[22] M. Burgos, A.C. Márquez-García, and A. Morales-Campoy. Determining jordan (triple) homomorphisms by invertibility preserving conditions. *Linear Multilinear Algebra*, 63(7):1294–1308, 2015.

[23] Chunyuan C. Deng and R. Liu. The existence and expressions of the inverse along operators B and C. *Oper. Matrices*, 12(4):1027–1042, 2018.

[24] G. Călugăreanu. On unit-regular rings. *Preprint*, 2010.

[25] G. Călugăreanu and P. Schultz. Modules with abelian endomorphism rings. *Bull. Aust. Math. Soc.*, 82(1):99–112, 2010.

[26] V.P. Camillo and D. Khurana. A characterization of unit regular rings. *Comm. Algebra*, 29(5):2293–2295, 2001.

[27] V.P. Camillo, D. Khurana, T.Y. Lam, W.K. Nicholson, and Y. Zhou. Continuous modules are clean. *J. Algebra*, 304(1):94–111, 2006.

[28] V.P. Camillo and H-P. Yu. Stable range one for rings with many idempotents. *Trans. Amer. Math. Soc.*, 347(8):3141–3147, 1995.

[29] S. Canu, C.S. Ong, and X. Mary. Splines with non positive kernels. In *More Progresses In Analysis*, pages 163–173. World Scientific, 2009.

[30] J-M. Cen. On the unified theory of matrix partial orders. *Shuxue Xuebao*, 22(1):139–143, 2005.

[31] A. Chambaz, E. Joly, and X. Mary. Targeted learning using adaptive survey sampling. In *Targeted Learning in Data Science*, pages 541–559. Springer, 2018.

[32] H. Chen. Extensions of Unit-Regular rings. *Comm. Algebra*, 32(6):2359–2364, 2004.

[33] H. Chen. Exchange rings with stable range one. *Czechoslovak Math. J.*, 57(2):579–590, 2007.

[34] H. Chen and M. Chen. On unit-regular ideals. *New York J. Math*, 9:295–302, 2003.

[35] J. Chen, Y. Ke, and D. Mosić. The reverse order law of the $(b, c)-$inverse in semigroups. *Acta Math. Hungar.*, 151(1):181–198, 2017.

[36] J. Chen, S. Xu, J. Benítez, and X. Chen. Rank equalities related to a class of outer generalized inverse. *Filomat*, 33(17):5611–5622, 2019.

[37] J. Chen, H. Zou, H. Zhui, and P. Patricio. The one-sided inverse along an element in semigroups and rings. *Mediterr. J. Math.*, 14(5):1–17, 2017.

[38] W. Chen. A question on strongly clean rings. *Comm. Algebra*, 34(7):2347–2350, 2006.

[39] M. Cho and G. Kantún-Montiel. Commuting outer inverses. *Adv. linear algebr. & matrix theor.*, 3(4):69, 2013.

[40] A.H. Clifford. A class of d-simple semigroups. *Am. J. Math.*, 75(3):547–556, 1953.

[41] Alfred H. Clifford and Gordon B. Preston. *The algebraic theory of semigroups. 2*. American Mathematical Soc., 1967.

[42] R.E. Cline. An application of representation for the generalized inverse of a matrix. *MRC Technical Report*, (592), 1965.

[43] A. Costa and B. Steinberg. The Schützenberger category of a semigroup. *Semigroup Forum*, 91(3):543–559, 2015.

[44] P. Crawley and B. Jónsson. Refinements for infinite direct decompositions of algebraic systems. *Pacific J. Math.*, 14(3):797–855, 1964.

[45] M. R. Croisot. Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples. *Ann. Sci. Ecole Norm. Sup.*, 70:361–379, 1953.

[46] H. D'Alarcao. Factorizable as a finiteness condition. *Semigroup Forum*, 20:281–282, 1980.

[47] A.J. Diesl. Nil clean rings. *J. Algebra*, 383:197–211, 2013.

[48] A.J. Diesl, S.J. Dittmer, and P.P. Nielsen. Idempotent lifting and ring extensions. *J. Algebra Appl.*, 15(06):1650112, 2016.

[49] D.S. Djordjević and P.S. Stanimirovic. On the generalized Drazin inverse and generalized resolvent. *Czechoslovak Mathematical Journal*, 51(3):617–634, 2001.

[50] M.P. Drazin. Pseudo-inverses in associative rings and semigroups. *Amer. Math. Monthly*, 65(7):506–514, 1958.

[51] M.P. Drazin. A class of outer generalized inverses. *Linear Algebra Appl.*, 436(7):1909–1923, 2012.

[52] M.P. Drazin. Commuting properties of generalized inverses. *Linear Multilinear Algebra*, 61(12):1675–1681, 2013.

[53] M.P. Drazin. Left and right generalized inverses. *Linear Algebra Appl.*, 510:64–78, 2016.

[54] M.P. Drazin. Bicommuting properties of generalized inverses. *Linear Multilinear Algebra*, 65(9):1770–1780, 2017.

[55] M.P. Drazin. Weighted (b, c)-inverses in categories and semigroups. *Comm. Algebra*, 48(4):1423–1438, 2020.

[56] M.P. Drazin. Hybrid (b, c)-inverses and five finiteness properties in rings, semigroups, and categories. *Comm. Algebra*, 49(5):2265–2277, 2021.

[57] A. Dress. On the decomposition of modules. *Bull. Am. Math. Soc.*, 75(5):984–986, 1969.

[58] R.E. Harte D.S. Djordjević and S.Č. Živković-Zlatanović. On simple permanence. *Quaest. Math.*, 38(4):515–528, 2015.

[59] X. Du. The adjoint semigroup of a ring. *Comm. Algebra*, 30(9):4507–4525, 2002.

[60] D. Easdown. Biordered sets come from semigroups. *J. Algebra*, 96(2):581–591, 1985.

[61] J. East and P.M. Higgins. Green's relations and stability for subsemigroups. *Semigroup Forum*, 101(1):77–86, 2020.

[62] G. Ehrlich. Unit-regular rings. *Port. Math.*, 27(4):209–212, 1968.

[63] G. Ehrlich. Units and one-sided units in regular rings. *Trans. Amer. Math. Soc.*, 216:81–90, 1976.

[64] Alberto Facchini. *Module theory: endomorphism rings and direct sum decompositions in some classes of modules*, volume 167. Birkhäuser, 2013.

[65] G. E. Fitz-Gerald. On inverses of products of idempotents in regular semigroups. *J. Aus. Math. Soc.*, 13(3):335–337, 1972.

[66] J. Fountain, G.M.S. Gomes, and V. Gould. A munn type representation for a class of E-semiadequate semigroups. *J. Algebra*, 218(2):693–714, 1999.

[67] J. Fountain and M. Petrich. Completely 0-simple semigroups of quotients. *J. Algebra*, 101(2):365–402, 1986.

[68] L. Fuchs. On a substitution property of modules. *Monatshefte für Mathematik*, 75(3):198–204, 1971.

[69] S. Garg, H.K. Grover, and D. Khurana. Perspective rings. *J. Algebra*, 415:1–12, 2014.

[70] Kenneth R Goodearl. *Von Neumann regular rings*. Krieger Pub Co, 1991.

[71] V. Gould. Straight left orders. *Stud. Sci. Math. Hungarica*, 30(3):355–374, 1995.

[72] V. Gould. Semigroups of left quotients: existence, straightness and locality. *J. Algebra*, 267(2):514–541, 2003.

[73] J.A. Green. On the structure of semigroups. *Ann. of Math.*, pages 163–172, 1951.

[74] Pierre Antoine Grillet. A coherence theorem on schützenberger groups. *J. Aust. Math. Soc.*, 21(2):129–143, 1976.

[75] R. Guralnick and C. Lanski. Pseudosimilarity and cancellation of modules. *Linear Algebra Appl.*, 47:111–115, 1982.

[76] A. Guterman, A. Herrero, and N. Thome. New matrix partial order based on spectrally orthogonal matrix decomposition. *Linear Multilinear Algebra*, 64(3):362–374, 2016.

[77] A. Guterman, L. Márki, and P. Shteyner. Ordering orders and quotient rings. *Semigroups, Categories, and Partial Algebras: ICSAA 2019, Kochi, India, December 9-12*, 345:1, 2021.

[78] A. Guterman, X. Mary, and P. Shteyner. Partial orders based on inverses along elements. *Journal of Mathematical Sciences*, 232(6):783–796, 2018.

[79] A. Guterman, X. Mary, and P. Shteyner. On hartwig–nambooripad orders. *Semigroup Forum*, 98(1):64–74, 2019.

[80] T.E. Hall. On regular semigroups. *J. algebra*, 24(1):1–24, 1973.

[81] T.E. Hall. Some properties of local subsemigroups inherited by larger subsemigroups. *Semigroup Forum*, 25(1):35–49, 1982.

[82] D. Handelman. Perspectivity and cancellation in regular rings. *J. Algebra*, 48(1):1–16, 1977.

[83] R.E. Harte. Spectral projections. *Irish Math. Soc. Newsletter*, 11:10–15, 1984.

[84] R.E. Harte. Fredholm, Weyl and Browder theory. *Math. Proc. R. Ir. Acad.*, pages 151–176, 1985.

[85] R.E. Hartwig. How to partially order regular elements. *Math. Japon.*, 25:1–13, 1980.

[86] R.E. Hartwig and J. Luh. A note on the group structure of unit regular ring elements. *Pacific J. Math.*, 71(2):449–461, 1977.

[87] R.E. Hartwig and J. Luh. On finite regular rings. *Pacific J. Math.*, 69(1):73–95, 1977.

[88] R.E. Hartwig and J. Luh. Decomposition of an orthogonally complete atomic unit regular ring. *unpublished report, Nov*, 1979.

[89] R.E. Hartwig and G.P.H. Styan. On some characterizations of the "star" partial ordering for matrices and rank subtractivity. *Linear Algebra Appl.*, 82:145–161, 1986.

[90] H. Heatherly. Adjoint groups and semigroups of rings. *Riv. Mat. Pura Appl*, 6:105–108, 1990.

[91] H. Heatherly. Adjoint groups of radical rings (**). *Riv. Mat. Univ. Parma*, 6(2):55–68, 1999.

[92] H. Heatherly and R.P. Tucci. The circle semigroup of a ring. *Acta Mathematica Hungarica*, 90(3):231–242, 2001.

[93] J.B. Hickey. A class of regular semigroups with regularity-preserving elements. *Semigroup Forum*, 81(1):145–161, 2010.

[94] J.B. Hickey and M.W. Lawson. Unit regular monoids. *Proc. R. Soc. Edinb. A: Math.*, 127(1):127–143, 1997.

[95] E. Hille and R.S. Phillips. *Functional analysis and semi-groups*, volume 31. American Mathematical Soc., 1996.

[96] C. Hollings. From right pp monoids to restriction semigroups: a survey. *Eur. J. Pure Appl. Math.*, 2(1):21–57, 2009.

[97] M. Hongan. Note on strongly regular near-rings. *Proc. Edinburgh Math. Soc.*, 29(3):379–381, 1986.

[98] John Mackintosh Howie. *Fundamentals of semigroup theory*, volume 12. Clarendon Oxford, New York, 1995.

[99] M. E. Hull and L.E. Renner. The centraliser of a semisimple element on a reductive algebraic monoid. *J. Algebra*, 218(1):117–125, 1999.

[100] N. Jacobson. The radical and semi-simplicity for arbitrary rings. *Am. J. Math.*, 67(2):300–320, 1945.

[101] Nathan Jacobson. *Structure of rings*, volume 37. American Mathematical Soc., 1956.

[102] B.W. Jordan and Y. Zaytman. On the bounded generation of arithmetic sl2. *Proc. Natl. Acad. Sci. USA*, 116(38):18880–18882, 2019.

[103] G. Kantún-Montiel. Natural generalized invertibility and prescribed idempotents. *arXiv preprint arXiv:1302.1480*, 2013.

[104] G. Kantún-Montiel. Outer generalized inverses with prescribed ideals. *Linear Multilinear Algebra*, 62(9):1187–1196, 2014.

[105] G. Kantún-Montiel and S. Djordjevic. Invertibility along an operator. *Oper. Matrices*, 11(2):347–354, 2017.

[106] Irving Kaplansky. *Fields and rings.* University of Chicago Press, 1972.

[107] Y. Ke, J. Chen, P. Stanimirović, and M. Ćirić. Characterizations and representations of outer inverse for matrices over a ring. *Linear Multilinear Algebra*, 69(1):155–176, 2021.

[108] Y. Ke, Z. Wang, and J. Chen. The (b, c)-inverse for products and lower triangular matrices. *J. Algebra Appl.*, 16(12):1750222, 2017.

[109] D. Khurana and T.Y. Lam. Clean matrices and unit-regular matrices. *J. Algebra*, 280(2):683–698, 2004.

[110] D. Khurana and T.Y. Lam. Rings with internal cancellation. *J. Algebra*, 284(1):203–235, 2005.

[111] D. Khurana, T.Y. Lam, and P.P. Nielsen. Two-sided properties of elements in exchange rings. *Algebr. Represent. Theory*, 18(4):931–940, 2015.

[112] D. Khurana, T.Y. Lam, and P.P. Nielsen. An ensemble of idempotent lifting hypotheses. *J. Pure Appl. Algebra*, 222(6):1489–1511, 2018.

[113] D. Khurana, T.Y. Lam, P.P. Nielsen, and J. Šter. Special clean elements in rings. *J. Algebra Appl.*, 19(11):2050208, 2020.

[114] D. Khurana, T.Y. Lam, and Z. Wang. Rings of square stable range one. *J. Algebra*, 338(1):122–143, 2011.

[115] D. Khurana and P.P. Nielsen. Perspectivity and von Neumann regularity. *Comm. Algebra*, pages 1–17, 2021.

[116] D. Khurana and P.P. Nielsen. Transitivity of perspectivity. *Algebr. Represent. Theory*, pages 1–7, 2021.

[117] D. Khurana, X.Mary, and P.P. Nielsen. Idempotent chains and bounded generation of SL_2. *submitted*.

[118] Mati Kilp, Ulrich Knauer, and Alexander V Mikhalev. Monoids, acts and categories, 2000.

[119] J.J. Koliha. A generalized Drazin inverse. *Glasg. Math. J.*, 38(3):367–381, 1996.

[120] J.J. Koliha. Isolated spectral points. *Proc. Amer. Math. Soc.*, 124(11):3417–3424, 1996.

[121] J.J. Koliha and P. Patrício. Elements of rings with equal spectral idempotents. *J. Aust. Math. Soc.*, 72(1):137–152, 2002.

[122] M. Krasner and L. Kaloujnine. Produit complet des groupes de permutations et problème d'extension de groupes. iii. *Acta Sci. Math.(Szeged)*, 14:69–82, 1951.

[123] E. Krishnan and C.S. Preethi. Unit-regular semigroups and rings. *rn*, 55:7, 2017.

[124] R.L. Kruse. On the circle group of a nilpotent ring. *The Amer. Math. Monthly*, 77(2):168–170, 1970.

[125] G. Kudryavtseva and V. Mazorchuk. On three approaches to conjugacy in semigroups. *Semigroup Forum*, 78(1):14–20, 2009.

[126] Tsit-Yuen Lam. Excursions in ring theory. *forthcoming*.

[127] Tsit-Yuen Lam. *Lectures on modules and rings*, volume 189. Springer Science & Business Media, 2012.

[128] Tsit-Yuen Lam. *A first course in noncommutative rings*, volume 131. Springer Science & Business Media, 2013.

[129] T.Y. Lam. A crash course on stable range, cancellation, substitution and exchange. *J. Algebra Appl.*, 3(03):301–343, 2004.

[130] T.Y. Lam and A.S. Dugas. Quasi-duo rings and stable range descent. *J. Pure Appl. Algebra*, 195(3):243–259, 2005.

[131] T.Y. Lam and W. Murray. Unit regular elements in corner rings. *Bull. HongKong Math. Soc.*, 1(1):61–65, 1997.

[132] M.V. Lawson. Rees matrix semigroups. *Proceedings of the Edinburgh Mathematical Society*, 33(1):23–37, 1990.

[133] M.V. Lawson. Semigroups and ordered categories. i. the reduced case. *J. Algebra*, 141(2):422–462, 1991.

[134] A. Leroy and J. Matczuk. Remarks on the Jacobson radical. *Rings, Modules and Codes*, 727:269–276, 2019.

[135] V. Loonis and X. Mary. Determinantal sampling designs. *Journal of Statistical Planning and Inference*, 199:60–88, 2019.

[136] R.A. Lubansky. Koliha-Drazin invertibles form a regularity. *Math. Proc. R. Ir. Acad.*, pages 137–141, 2007.

[137] S.Y. Ma, X.M. Ren, and Y. Yuan. Completely j-simple semigroups. *Acta Math Sinica (Chin Ser)*, 54(4):643–650, 2011.

[138] G. Marks. A criterion for unit-regularity. *Acta Mathematica Hungarica*, 111(4), 2006.

[139] X. Mary. n-chained semigroups and $n/2$-perspective modules and rings. *to appear in Comm. Algebra.*

[140] X. Mary. Rings with transitive chaining of idempotents. *submitted.*

[141] X. Mary. Special clean and perspective elements and perspective rings. *to appear in J. Algebra Appl.*

[142] X. Mary. *Hilbertian subspaces, subdualities and applications*. PhD thesis, INSA de Rouen, 2003.

[143] X. Mary. Theory of subdualities. *Journal d'Analyse Mathématique*, 97(1):203–241, 2005.

[144] X. Mary. Moore-Penrose inverse in Kreĭn spaces. *Integral Equ. Oper. Theory*, 60(3):419–433, 2008.

[145] X. Mary. On the converse of a theorem of Harte and Mbekhta: Erratum to "on generalized inverses in C^*-algebras". *Studia Mathematica*, 184:149–151, 2008.

[146] X. Mary. On generalized inverses and Green's relations. *Linear Algebra Appl.*, 434(8):1836–1844, 2011.

[147] X. Mary. Natural generalized inverse and core of an element in semigroups, rings and Banach and operator algebras. *Eur. J. Pure Appl. Math.*, 5(2):160–173, 2012.

[148] X. Mary. Classes of semigroups modulo green's relation $\mathcal{H}$. *Semigroup Forum*, 88(3):647–669, 2014.

[149] X. Mary. Reverse order law for the group inverse in semigroups and rings. *Comm. Algebra*, 43(6):2492–2508, 2015.

[150] X. Mary. On $(E, \widetilde{H}_E)$-abundant semigroups and their subclasses. *Semigroup Forum*, 94(3):738–776, 2017.

[151] X. Mary. Weak inverses of products – Cline's formula meets Jacobson lemma. *J. Algebra Appl.*, 2017.

[152] X. Mary. A local structure theorem for stable, $\mathcal{J}$–simple semigroup biacts. 99(3):724–753, 2019.

[153] X. Mary. Characterizations of clean elements by means of outer inverses in rings and applications. *J. Algebra Appl.*, 19(7):2050134, 2020.

[154] X. Mary. (b, c)-inverse, inverse along and element, and the Schützenberger category of a semigroup. *Categories Gen. Algebraic Struct. with Appl.*, 15(1):255–272, 2021.

[155] X. Mary, D. De Brucq, and S. Canu. Sous-dualités et noyaux (reproduisants) associés. *C. R. Acad. Sci*, 336(11):949–954, 2003.

[156] X. Mary and P.P. Nielsen. IC rings and transitivity of perspectivity. *J. Algebra Appl.*, page 2250151, 2021.

[157] X. Mary and P. Patrício. The inverse along a lower triangular matrix. *Applied Math. Comp.*, 219(3):886–891, 2012.

[158] X. Mary and P. Patrício. Generalized inverses modulo $\mathcal{H}$ in semigroups and rings. *Linear Multilinear Algebra*, 61(8):1130–1135, 2013.

[159] X. Mary and P. Patrício. The group inverse of a product. *Linear Multilinear Algebra*, 64(9):1776–1784, 2016.

[160] X. Mary and P. Patrício. Group-regular rings. *Filomat*, 33(11):3551–3560, 2019.

[161] X. Mary and P. Patrício. Characterizations of special clean elements and applications. *Filomat*, 34(14):4847–4860, 2020.

[162] R. Mazurek, P.P. Nielsen, and M. Ziembowski. Commuting idempotents, square-free modules, and the exchange property. *J. Algebra*, 444:52–80, 2015.

[163] D.B. McAllister. Some covering and embedding theorems for inverse semigroups. *J. Aust. Math. Soc.*, 22(2):188–211, 1976.

[164] R.B. McFadden. Unit-regular orthodox semigroups. *Glasg. Math. J.*, 25(2):229–240, 1984.

[165] P. Menal and J. Moncasi. Lifting units in self-injective rings and an index theory for Rickart c^*-algebras. *Pacific J. Math.*, 126(2):295–329, 1987.

[166] D.D. Miller and A.H. Clifford. Regular $\mathcal{D}$-classes in semigroups. *Tran. Amer. Math. Soc.*, 82(1):270–280, 1956.

[167] S.K. Mitra. The minus partial order and the shorted matrix. *Linear Algebra Appl.*, 83:1–27, 1986.

[168] S.K. Mitra. On group inverses and the sharp order. *Linear Algebra Appl.*, 92:17–37, 1987.

[169] S.K. Mitra. Matrix partial orders through generalized inverses: unified theory. *Linear Algebra Appl.*, 148:237–263, 1991.

[170] S.K. Mitra and R.E. Hartwig. Partial orders based on outer inverses. *Linear Algebra Appl.*, 176:3–20, 1992.

[171] H. Mitsch. A natural partial order for semigroups. *Proc. Amer. Math. Soc.*, 97(3):384–388, 1986.

[172] G.S. Monk. A characterization of exchange rings. *Proc. Amer. Math. Soc.*, 35(2):349–353, 1972.

[173] E.H. Moore. On the reciprocal of the general algebraic matrix. *Bull. Am. Math. Soc.*, 26:394–395, 1920.

[174] D. Mosić. A note on Cline's formula for the generalized Drazin inverse. *Linear Multilinear Algebra*, 63(6):1106–1110, 2015.

[175] D. Mosić, P. S. Stanimirović, J. K. Sahoo, R. Behera, and V. N. Katsikis. One-sided weighted outer inverses of tensors. *J. Comput. Appl. Math.*, 388:113293, 2021.

[176] WD Munn. Semigroups satisfying minimal conditions. *Glasg. Math. J.*, 3(3):145–152, 1957.

[177] K.S.S. Nambooripad. Structure of regular semigroups, I fundamental regular semigroups. *Semigroup forum*, 9(1):354–363, 1974.

[178] K.S.S. Nambooripad. The natural partial order on a regular semigroup. *Proc. Edinb. Math. Soc.*, 23(3):249–260, 1980.

[179] J. Von Neumann. *Continuous geometry.* Princeton University Press, 2016.

[180] W.K. Nicholson. Lifting idempotents and exchange rings. *Trans. Amer. Math. Soc.*, 229:269–278, 1977.

[181] W.K. Nicholson. On exchange rings. *Comm. Algebra*, 25(6):1917–1918, 1997.

[182] W.K. Nicholson. Strongly clean rings and Fitting's lemma. *Comm. Algebra*, 27(8):3583–3592, 1999.

[183] W.K. Nicholson and Y. Zhou. Clean general rings. *J. Algebra*, 291(1):297–311, 2005.

[184] P.P. Nielsen. Countable exchange and full exchange rings. *Comm. Algebra*, 35(1):3–23, 2006.

[185] P.P. Nielsen and J. Šter. Connections between unit-regularity, regularity, cleanness, and strong cleanness of elements and rings. *Trans. Amer. Math. Soc.*, 370(3):1759–1782, 2018.

[186] Robert H Oehmke. The semigroup of a strongly connected automaton. *Semigroup Forum*, 15(1):351–356, 1977.

[187] C.S. Ong, X. Mary, S. Canu, and A.J. Smola. Learning with non-positive kernels. In *Proceedings of the twenty-first international conference on Machine learning*, page 81, 2004.

[188] F. Pastijn. The biorder on the partial groupoid of idempotents of a semigroup. *J. Algebra*, 65(1):147–187, 1980.

[189] R. Penrose. A generalized inverse for matrices. *Math. Proc. Cambridge Philos. Soc.*, 51(3):406–413, 1955.

[190] M. Petrich. A structure theorem for completely regular semigroups. *Proc. Amer. Math. Soc.*, 99(4):617–622, 1987.

[191] M. Petrich. Onesided inverses for semigroups. *Acta Math. Univ. Comenianae*, 75(1):1–19, 2006.

[192] M. Petrich. Characterizing pure, cryptic and clifford inverse semigroups. *Czechoslov. Math. J.*, 64(4):1099–1112, 2014.

[193] M. Petrich. A semilattice of varieties of completely regular semigroups. *Math. Bohem.*, 145(1):1–14, 2020.

[194] Mario Petrich. *Introduction to semigroups.* Merrill, Ohio, 1973.

[195] M.S. Putcha. Regular linear algebraic monoids. *Trans. Amer. Math. Soc.*, 290(2):615–626, 1985.

[196] D S. Rakić and D.S. Djordjević. Partial orders in rings based on generalized inverses–unified theory. *Linear Algebra Appl.*, 471:203–223, 2015.

[197] D.S. Rakić, N. Dinčić, and D.S. Djordjević. Group, Moore–Penrose, core and dual core inverse in rings with involution. *Linear Algebra Appl.*, 463:115–133, 2014.

[198] A. Rakotomamonjy, X. Mary, and S. Canu. Wavelet kernel and rkhs. *Proc. of Statistical Learning: Theory and Applications, Paris*, pages 85–88, 2002.

[199] A. Rakotomamonjy, X. Mary, and S. Canu. Non-parametric regression with wavelet kernels. *Applied Stochastic Models in Business and Industry*, 21(2):153–163, 2005.

[200] X. Ren, K.P. Shum, and Y.Q. Guo. A generalized clifford theorem of semigroups. *Science China Mathematics*, 53(4):1097–1101, 2010.

[201] V.K. Sreeja and A.R. Rajan. Construction of certain unit regular orthodox submonoids. *Southeast Asian Bull. Math.*, 38(6), 2014.

[202] V.K. Sreeja and A.R. Rajan. Some properties of regular monoids. *Southeast Asian Bull. Math.*, 39(6), 2015.

[203] P. S. Stanimirović, M. Ćirić, V. N. Katsikis andC. Li, and H. Ma. Outer and (b, c) inverses of tensors. *Linear Multilinear Algebra*, 68(5):940–971, 2020.

[204] Benjamin Steinberg. A theory of transformation monoids: combinatorics and representation theory. *the electronic journal of combinatorics*, 17(1):R164, 2010.

[205] V.V. Vagner. Generalized groups. *Dokl. Akad. Nauk SSSR*, 84(6):1119–1122, 1952.

[206] L. Vas. Cancellation properties of nonunital rings. graded clean and graded exchange leavitt path algebras. *arXiv preprint arXiv:2106.12031*, 2021.

[207] L.N. Vaserstein. Bass's first stable range condition. *J. Pure Appl. Algebra*, 34(2-3):319–330, 1984.

[208] L. Wang and D. Mosić. The one-sided inverse along two elements in rings. *Linear Multilinear Algebra*, pages 1–13, 2019.

[209] L. Wang, D. Mosić, and Y. Gao. Right core inverse and the related generalized inverses. *Comm. Algebra*, 47(11):4749–4762, 2019.

[210] L. Wang, D. Mosić, and H. Yao. The commuting inverse along an element in semigroups. *Comm. Algebra*, pages 1–11, 2021.

[211] W. Wang, S. Xu, and J. Benítez. Rank equalities related to the generalized inverses $A^{\|(b_1,c_1)}$, $D^{\|(b_2,c_2)}$ of two matrices A and D. *Symmetry*, 11(4):539, 2019.

[212] Z. Wang. A class of Drazin inverses in rings. *Filomat*, 31(6):1781–1789, 2017.

[213] R. Warfield. Exchange rings and decompositions of modules. *Math. Ann.*, 199(1):31–36, 1972.

[214] R.B. Warfield. A Krull-Schmidt theorem for infinite sums of modules. *Proc. Amer. Math. Soc.*, 22(2):460–465, 1969.

[215] C. Wu and J. Chen. On (b, c)-inverses and (c, b)-inverses. *Comm. Algebra*, pages 1–11, 2021.

[216] Y. Wu, G. Tang, G. Deng, and Y. Zhou. Nil-clean and unit-regular elements in certain subrings of $\mathbb{M}_2(\mathbb{Z})$. *Czechoslovak Math. J.*, 69(1):197–205, 2019.

[217] X.Fan, Q. Chen, and X. Kong. Weak inverses modulo green's relation $\mathcal{H}$ on e-inversive and group-closed semigroups. *Semigroup Forum*, 92(3):691–711, 2016.

[218] S. Xu and J. Benítez. Existence criteria and expressions of the $(b, c)-$inverse in rings and their applications. *Mediterr. J. Math.*, 15(1):14, 2018.

[219] XS. Xu, J. Chen, J. Benítez, and D.Wang. Centralizer's applications to the (b, c)-inverses in rings. *Rev. R. Acad. Cienc. Exactas Fıs. Nat. Ser. A Math. RACSAM*, 113(3):1739–1746, 2019.

[220] M. Yamada and K. Shoji. On h-compatible quasi-orthodox semigroups. *Mem. Fac. Sci. Shimane Univ*, 13:1–9, 1979.

[221] Y. Yuan, C. Gong, and S. Ma. The structure of u-superabundant semigroups and the translational hull of completely j-simple semigroups. *Adv. Math.(China)*, 43:35–47, 2014.

[222] H. Zhu. Further results on several types of generalized inverses. *Comm. Algebra*, 46(8):3388–3396, 2018.

[223] H. Zhu, J. Chen, and P. Patrício. Further results on the inverse along an element in semigroups and rings. *Linear Multilinear Algebra*, 64(3):393–403, 2016.

[224] H. Zhu, J. Chen, and P. Patrício. Reverse order law for the inverse along an element. *Linear Multilinear Algebra*, 65(1):166–177, 2017.

[225] H. Zhu, J. Chen, P. Patrício, and X. Mary. Centralizer's applications to the inverse along an element. *Applied Math. Comp.*, 315:27–33, 2017.

[226] H. Zhu, J. Chen, P. Patrício, and Y. Zhang. The inverse along a product and its applications. *Linear Multilinear Algebra*, 64(5):834–841, 2016.

[227] H. Zhu and P. Patrício. A note on clean elements and inverses along an element. *Filomat*, 32(12):4285–4290, 2018.

[228] H. Zhu and P. Patrício. Several types of one-sided partial orders in rings. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 113(4):3177–3184, 2019.

[229] G. Zhuang, J. Chen, and J. Cui. Jacobson's lemma for the generalized Drazin inverse. *Linear Algebra Appl.*, 436(3):742–746, 2012.