E-SOLID RINGS, STRONGLY IC RINGS AND THE JACOBSON RADICAL

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Abstract. In this article, we prove that the well-known semigroup property of being E-solid is actually equivalent to abelianness modulo the Jacobson radical in the case of (possibly non-unital) non-regular rings. We also give various other characterizations of such rings.

This article is dedicated to André Leroy on his retirement.

1. Introduction

The coexistence of two compatible algebraic structures on a ring, the multiplicative (semigroup) one and the additive one confers a ring some strong rigidity relative to mere semigroups. For instance, it is well-known that strongly regular rings are exactly those rings where each element admits a unique reflexive inverse (inverse rings), and that they also coincide with abelian regular rings. By contrast, completely regular semigroups, inverse semigroups and Clifford semigroups form distinct classes of semigroups. In this note, we consider neither regular not necessarily unital rings whose multiplicative semigroup is E-solid, and prove that they are characterized by their idempotents being central modulo the Jacobson radical. We also provide the reader with various other characterizations of such rings in terms of the Jacobson radical, reflexive inverses, chains of idempotents or square stable range one. In particular, we prove that unital E-solid rings coincide with the strongly IC rings of Khurana, Lam and Wang [10], the J-abelian rings (see [5], [12] and also [11]), the strongly perspective rings [8], [12], the superspective rings [21] or the 1/2-perspective rings of the present author [18].

First, we quickly review a few of the basics concepts here. In the sequel, rings will be associative but not necessarily unital.

An element x of a ring R is said to be *regular* if there exists some $y \in R$ such that $xyx = x$. The element y is called an *inner inverse* (formerly *quasi-inverse*) of x. When the pair of equalities $xyx = x$ and $yxy = y$ hold, we say that x and y are *reflexive inverses* of each other, or that (x, y) forms a *reflexive pair*. We denote by $V(x)$ the set of reflexive inverses of x. It is well known that every regular element has a reflexive inverse (take $y' = yxy$).

An element $x \in R$ is *strongly regular* if it admits a commuting inner (equiv. reflexive) inverse, and *unit-regular* if it admits an inner inverse that is also a unit. A ring such that every element is regular (resp. strongly regular, unit-regular) is called a *regular ring* (resp. *strongly regular ring*, *unit-regular ring*).

An idempotent $e \in \mathrm{idem}(R)$ is *left semicentral* if $(\forall x \in R)\ xe = exe$. Two idempotents $e, f \in \mathrm{idem}(R)$ are *isomorphic* (denoted by $e \simeq f$) if eR and fR are isomorphic submodules of R_R, iff $e = xy$, $f = yx$ for some $x, y \in R$. Moreover, we can always choose $y \in V(x)$. The idempotents e and f are *left (resp. right) associates*, and we write $e \sim_\ell f$ (resp. $e \sim_r f$)

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if $Re = Rf$ or equivalently if $ef = e$ and $fe = f$ (resp. $eR = fR$ or equivalently $ef = f$ and $fe = e$). Finally, any two idempotents e, f are weakly (resp. strongly) 2-chained if $e \sim_r g \sim_\ell f$ for some $g \in \text{idem}(R)$ or (resp. and) $e \sim_\ell h \sim_r f$ for some $h \in \text{idem}(R)$.

Since we consider non necessarily unital ring, we recall the characterization of the Jacobson radical in terms of quasi-regular elements:

$$J(R) = \{s \in R | (\forall p \in R)\, sp \text{ is right quasiregular}\}$$

where an element $x \in R$ is right quasiregular if $x + y - xy = 0$ for some $y \in R$. As is well-known, the elements of $J(R)$ are themselves both left and right quasiregular [13].

The previous purely multiplicative notions also make sense in semigroups. Traditionally, idempotents of a semigroup S are however denoted by $E(S)$ and strongly regular elements are termed *completely regular*. We finally recall Clifford and Miller's theorem, interpreted in terms of associated idempotents.

Theorem 1.1 ([20, Theorem 3])**.** *Let S be a semigroup and let $e, f \in E(S)$ be isomorphic idempotents ($e = ab, f = ba$ for some $a, b \in S$ with $b \in V(a)$). Then $e \in Rfe, f \in feR$ iff $e \sim_r h \sim_\ell f$ for some $h \in E(S)$ iff a is completely regular.*

2. E-solid rings

2.1. **Classes of semigroups.** Apart groups, the following classes of regular semigroups are among the most notable ones:

(1) Inverse semigroups (each element admits a unique reflexive inverse; equiv. regular semigroups whose idempotents commute);
(2) Completely regular semigroups (each element admits a commuting inner inverse);
(3) Clifford semigroups (each element admits a unique reflexive inverse, that moreover commutes with the element; equiv. regular semigroups with central idempotents);
(4) Orthodox semigroups (regular semigroups whose set of idempotents forms a subsemigroup);
(5) Locally inverse semigroups (each local submonoid is inverse);
(6) E-solid regular semigroups (the idempotent generated subsemigroup $\langle E(S)\rangle$ is completely regular).

Inverse semigroups appear naturally in connections with other areas of mathematics to model partial symmetries [15], while completely regular semigroups proved one of the most elegant classes of semigroups, with a well established structure. Moreover, these two classes carry a natural inversion, and with this inversion they form *varieties* of unary semigroups. The other classes form e-varieties of regular semigroups [6], which allows an equational description (by a generalization of Birkhoff theorem). Inverse semigroups are orthodox, and orthodox semigroups are E-solid regular.

Regarding non-regular semigroups, we say that:

(1) S is an *E-semigroup* if $E(S)$ is a subsemigroup;
(2) S is *E-commutative* if $E(S)$ is commutative;
(3) S is *E-solid* if for any three idempotents $e, f, g \in E(S)$, if $e \sim_r g \sim_\ell f$ then there exists $h \in E(S)$ such that $e \sim_\ell h \sim_r f$, and dually if $e \sim_\ell g \sim_r f$ then there exists $h \in E(S)$ such that $e \sim_r h \sim_\ell f$. In terms of chains of idempotents, S is E-solid iff any weakly 2-chained idempotents are strongly 2-chained.

2.2. **The ring case.** For any ring $R = (R, +, .)$, its multiplicative part $(R, .)$ forms a semigroup. We say that the ring $R = (R, +, .)$ is inverse (resp. strongly regular, resp. Clifford, resp. E-solid) if $(R, .)$ is an inverse (resp. completely regular, resp. Clifford, resp. E-solid) semigroup.

As recalled in the introduction, in the case of regular rings, a remarkable collapsing occurs: strongly regular rings and inverse rings coincide, and are therefore Clifford rings, more usually called *abelian regular rings* (regular rings whose idempotents are central). It is also known that they coincide with orthodox rings [24]. They admit many other characterizations, among others they are the reduced regular rings, the regular subdirect products of division rings, or the regular rings such that $(\forall e, f \in \text{idem}(R))\ ef = 0 \Rightarrow fe = 0$. In 1997, J. Loyola proved that they also coincide with E-solid regular rings [16]. The proofs therein rest heavily upon regularity. Actually, a little more is true: such rings can also be characterized locally (by considering their corner rings $eRe, e \in \text{idem}(R)$) as the *locally inverse rings* or the *locally E-solid* regular rings [7, Proposition 3.1].

In case of non-regular rings, the first result was probably the characterization of unital rings whose regular elements are strongly regular by Khurana, Lam and Wang [10] in terms of elements of *square stable range one* (an element $a \in R$ has (right) square stable range one if $aR + bR = R$ (for any $b \in R$) implies that $a^2 + by$ is a unit for some $y \in R$). Such rings are also called *strongly IC rings* by analogy with rings with internal cancellation (IC rings [9]) that are characterized by their regular elements being unit-regular. Otherwise, few results were known in relations with the previous notions until recently and the obtainment by Khurana, Lam, Nielsen, Patricio and the present author of different results based on the Jacobson radical [11, Theorem 3.13], [12, Theorem 3.11], [17, Theorems 2.1 and 2.2], [18, Proposition 4.16 and Corollary 4.17], [19, Theorem 6.3]. Among these results, while a relation between strongly regular elements and the Jacobson radical was already observed for instance in [19], it is Khurana and Nielsen [11] that discovered the full strength of the relation between strongly IC rings and the Jacobson radical, and the link with chains of idempotents. They notably established equivalences instead of mere implications (many of these equivalences are recalled below). As such, [11] served as a catalyst for much of the subsequent research on the subject, including the present article. To try to be thorough on this topic, let us add two comments. First, rings having idempotents central modulo their Jacobson radical are called *J-abelian* in [5] and [12], and we will use this terminology afterwards. Second, those rings are determined by a very nice property in terms of the direct summands of the module R_R, as proved independently in [11, Theorem 3.17], [12, Theorem 3.11] or [18, Corollary 4.10]: R is J-abelian iff for any two isomorphic direct summands A, B of R_R, any direct complement of A is a direct complement of B (quoting [11] "isomorphic direct summands share all their complements"). For this reason, they are also called *strongly perspective rings* [8], [12], *1/2-perspective rings* [18] or *superspective rings* [21].

We collect all these known results below (unital ring case). While most of the proofs carry verbatim to the non-unital ring case, we nevertheless provide the reader with a general scheme to prove they remain true in the non-unital ring case.

Theorem 2.1 ([10], [11], [12], [17])**.** *Let R be ring, and consider the following statements:*
(1) *regular elements of R are strongly regular (R is strongly IC);*
(2) *regular elements of R have (right) square stable range one;*

(3) *strong* 2*-chaining is transitive;*
(4) *weak* 2*-chaining is transitive;*
(5) *isomorphic idempotents are strongly* 2*-chained;*
(6) *isomorphic idempotents are weakly* 2*-chained;*
(7) *isomorphic idempotents of* R *are equal modulo the Jacobson radical;*
(8) *idempotents of* R *are central modulo the Jacobson radical (*R *is J-abelian).*

Then all these statements are equivalent in case R *is unital. Moreover, the equivalences (without statement* (2)*) remain valid in non-unital rings.*

We propose here to prove the non-unital case by using the classical unitization (Dorroh extension) of a ring as a general methodology. Indeed, let R be a ring and $\hat{R} = R \oplus \mathbb{Z}$ be its classical Dorroh extension, with product $(a, n)(b, p) = (ab + pa + nb, np)$. Then idempotents of $\hat{R}$ are of the form $e_0 = (e, 0)$ or $e_1 = (-e, 1) = 1 - e_0$, where $e \in \text{idem}(R)$. Moreover, isomorphic (in particular associate) idempotents in $\hat{R}$ share the same second coordinate. It follows that chains of idempotents of $\hat{R}$ either live in the ideal R of $\hat{R}$, or so does the chain of their complementary idempotents (if $e_1 \sim_r f_1$ then $1 - e_1 \sim_\ell 1 - f_1$ for any two idempotents e_1, f_1 of a unital ring). Then almost each of the equivalences follow from the previously proved unital results, and this general structural relationship. Below, we derive all but one equivalence (in the non-unital case) by this method. The remaining one cannot be proven this way because isomorphic idempotents in $\hat{R}$ of the form $e_1 = (-e, 1), f_1 = (-f, 1)$ need not have e and f isomorphic in R in general. The remaining equivalence will thus be proved independently in Lemma 2.2. Some equivalences will also be proven directly in Theorem 2.3.

Proof of Theorem 2.1. We prove that (1) $\iff$ (3) $\iff$ (4) $\iff$ (5) $\iff$ (6) $\iff$ (8) $\Rightarrow$ (7) in the non-unital case. First, we observe that the following implications are straightforward even in the non-unital case: (5) $\Rightarrow$ (3), (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (4). Also (1) $\iff$ (5) by Miller and Clifford's Theorem 1.1.
Let $\hat{R} = R \oplus \mathbb{Z}$ and $e \in \text{idem}(R)$ be central modulo $J(R)$. As is well-known, $J(\hat{R}) = (J(R), 0)$ so that for any $x \in R$ and $n \in \mathbb{Z}$, the commutator $[e_i; (x, n)] = ((-1)^i(ea - ae), 0) \in J(\hat{R})$ ($i \in \{0, 1\}$), and $e_0, e_1 \in \text{idem}(\hat{R})$ are central modulo $J(\hat{R})$.
Second, if $e \in \text{idem}(R)$ is associate to $f \in \text{idem}(\hat{R})$, then $f \in R$ as R is an ideal of $\hat{R}$. By the same argument, $a \in R$ is strongly regular in $\hat{R}$ iff it is strongly regular in R.
We conclude that (8) $\Rightarrow$ (1), (3), (4), (5), (6), (7) in non-unital rings.
We finally prove that (3) $\Rightarrow$ (8) (the proof of (4) $\Rightarrow$ (8) is similar). Assume (3) and let $e_i, f_j, g_k \in \text{idem}(\hat{R})$ ($i, j, k \in \{0, 1\}$) be such that e_i, f_j and f_j, g_k are strongly 2-chained in $\hat{R}$. Suppose first that $i = 0$. Then $j = k = 0$ (associate idempotents in $\hat{R}$ share the same second coordinate) and e, f and f, g are strongly 2-chained in R. By the transitivity assumption, e and g are strongly 2-chained in R hence e_0, g_0 are strongly 2-chained in $\hat{R}$. Suppose now that $i = 1$. Then $j = k = 1$ and $e_1 = 1 - e_0, f_1 = 1 - f_0$ and $f_1 = 1 - f_0, g_1 = 1 - g_0$ are strongly 2-chained, so that their complementary idempotents are also strongly 2-chained. Thus as previously, e_0, g_0 are strongly 2-chained in $\hat{R}$ and so are their complementary idempotents. □

In passing, we have proved that, if R has central idempotents modulo its Jacobson radical, then so does its Dorroh extension $\hat{R} = R \oplus \mathbb{Z}$. We will return to the Dorroh extension, unitizations and square stable range 1 at the end of this section.

The remaining implication (7) ⇒ (5) is the content of Lemma 2.2. As pointed out by Pace Nielsen, this result is actually a specialization of [2, Proposition 2.4].

Lemma 2.2. *Let R be a ring, and $e, f \in \operatorname{idem}(R)$ be such that $e \equiv f\,(mod\ J(R))$. Then e and f are strongly 2-chained.*

Proof. By duality, we only have to prove that e and f are left 2-chained. So assume that $f = e - j$ for some $j \in J(R)$, and let $q \in R$ be such that $j + q - jq = 0$. Then

$$ef(f - q) = ef - efq = (e - ej) - (eq - ejq) = e - e(j + q - jq) = e.$$

This proves that $e \in efR$. By duality $f \in Ref$ and we conclude by Miller and Clifford's Theorem 1.1. □

Our main theorem proves that E-solid rings are exactly J-abelian rings. Moreover, they can also be characterized by uniqueness of reflexive inverses modulo the Jacobson radical, commutation of idempotents modulo the Jacobson radical or by product of idempotents being idempotents modulo the Jacobson radical.

Theorem 2.3. *Let R be a (possibly non-unital) ring. Then the following statements are equivalent:*

(1) *reflexive inverses are unique modulo the Jacobson radical;*
(2) *idempotents of R are central modulo the Jacobson radical (R is J-abelian);*
(3) *idempotents commute modulo the Jacobson radical;*
(4) *for all $e, f \in \operatorname{idem}(R)$, $ef \equiv f\,(mod\ J(R)) \iff fe \equiv f\,(mod\ J(R))$;*
(5) *weakly 2-chained idempotents are equal modulo the Jacobson radical;*
(6) *isomorphic idempotents are equal modulo the Jacobson radical;*
(7) *isomorphic idempotents are strongly 2-chained;*
(8) *regular elements are strongly regular (R is strongly IC);*
(9) *products of idempotents are idempotents modulo the Jacobson radical;*
(10) *R is E-solid (weakly 2-chained idempotents are strongly 2-chained).*

(they are also equivalent with statements (3), (4), (6) of Theorem 2.1.)

Proof. The implications (2) ⇒ (3) ⇒ (4) and (3) ⇒ (9) are straightforward, (6) ⇒ (7) by Lemma 2.2, (7) ⟺ (8) by Theorem 1.1 and (7) ⇒ (10), (6) ⇒ (5) as associate idempotents are isomorphic (and isomorphy is transitive). We prove that (4) ⇒ (5) ⇒ (1) ⇒ (2) and that (9) ⇒ (10) ⇒ (2) ⇒ (6).

(4) ⇒ (5) : Assume (4) and let $e, f, g \in E(S)$ so that $e \sim_r g \sim_\ell f$ (the other case $e \sim_\ell g \sim_r f$ is dual). As $e = ge, eg = g = gf$, $fg = f$ then by (4) $e \equiv g \equiv f\,(mod\ J(R))$.

(5) ⇒ (1) : Let $a \in R$ be a regular element and let $b, b' \in V(a)$. Then $ab \sim_r ab'$ hence $ab \equiv ab'\,(mod\ J(R))$, so that $b \equiv bab'\,(mod\ J(R))$. Dually $ba \sim_\ell b'a$ hence $ba \equiv b'a\,(mod\ J(R))$, so that $bbab' \equiv b'\,(mod\ J(R))$. Finally $b \equiv b'\,(mod\ J(R))$.

(1) ⇒ (2) : Let $e \in \operatorname{idem}(R)$ and $x \in R$. Pose $f = e + (ex - exe)$. Then $f \in V(e)$ hence $f \equiv e\,(mod\ J(R))$ by assumption or equivalently, $f - e \equiv ex - exe \equiv 0\,(mod\ J(R))$. This proves that e is right semicentral modulo $J(R)$ and we conclude by duality.

(9) ⇒ (10) : Assume (9) and let $e, f, g \in \operatorname{idem}(R)$ be such that $e \sim_r g \sim_\ell f$ (the other case $e \sim_\ell g \sim_r f$ is dual). As $e = ge, eg = g = gf$ and $fg = f$ then $ef = efg = efgf = efegf = gefegf = gfefegf$. By (9), $(fe)^2 \equiv fe\,(mod\ J(R))$ so that $ef \equiv gfegf \equiv egf \equiv g\,(mod\ J(R))$. Recall that an ideal I of R is *enabling* [1] if for any $x \in R$ and $e \in \operatorname{idem}(R)$, $x \equiv e\,(mod\ I)$ implies the existence of $f \in xR$ such that $f^2 = f$ and $f \equiv e\,(mod\ I)$. Now

the Jacobson radical is always an enabling ideal by [1, Proposition 5] (or [2, Corollary 3.2]) so that we can use [2, Proposition 3.4]: there exists an idempotent $h \in fRe$ such that $h \equiv fge\,(mod\ J(R))$. Moreover, as $e, f \in \text{idem}(R)$ and $ge = e, fg = f$ then h satisfies additionally that $e \sim_\ell h \sim_r f$. Finally e and f are strongly 2-chained and R is E-solid.

(10) ⇒ (2) : Assume that R is E-solid and let $e \in R$. We prove that e is right semicentral modulo $J(R)$ ($ex - exe \in J(R)$ for all $x \in R$). Let any $x, p \in R$ and pose $s = ex - exe$. We want to prove that sp is right quasi-regular. We decompose $sp = spe + (sp - spe)$. Let $t = pe - epe$ (so that $spe = st$). Then $e+s, e+t$ are idempotents and $e+s \sim_r e \sim_\ell e+t$. As R is E-solid then $e+s \sim_\ell h \sim_r e+t$ for some idempotent h and by Clifford and Miller's theorem, $(e+s)(e+t) = e+st$ is a unit in eRe. Let q be its inverse in eRe, and let $z = -qsp$. As $eq = q$ and $spe = st$ then $sp + z - spz = sp - qsp + spqsp = sp - qsp + speqsp = sp - qsp + stqsp$ and as $es = s$ and $(e + st)q = e = eq + stq = q + stq$ then $sp = esp = qsp + stqsp$, so that finally $sp + z - spz = 0$. It follows that sp is right-quasiregular, and by universality of $p \in R$, $s \in J(R)$. Since $x \in R$ is also arbitrary e is right semicentral modulo $J(R)$. The conclusion follows by duality.

(2) ⇒ (6) : Let $e = ab, f = ba$ be isomorphic idempotents in R with $b \in V(a)$. As $aba = a$ and $ab \in \text{idem}(R)$ then $a \equiv a^2b \equiv ba^2\,(mod\ J(R))$, and multiplying by b on the right and on the left gives $ab \equiv ba^2b \equiv ba\,(mod\ J(R))$. □

The proof of (9) ⇒ (10) relies on deep results of [2]. We can avoid them and prove directly that (9) ⇒ (8). In due course, we will however need the following result due to Lam that has not appeared yet [12, Theorem 4.3]: A ring R is a strongly IC ring iff $eR(1 - e)Re \subseteq J(R)$ for every idempotent $e \in \text{idem}(R)$ (and this works also for non-unital rings). Then assume (9) and let $e \in \text{idem}(R), x, y \in R$. Pose $s = ex - exe$ and $t = ye - eye$. As $s^2 = se = 0 = et = t^2$ and $es = s, te = t$ then $e + s$ and $e + t$ are idempotents and by (9) $(e + t)(e + s) = e + s + t + ts$ is idempotent modulo $J(R)$. But $e\left((e + t)(e + s)\right)^2 e = (e + s)(e + t) = e + st$ so that passing to the equivalences classes we obtain $e + st \equiv e\left((e + t)(e + s)\right)^2 e \equiv e\left((e + t)(e + s)\right) e \equiv e\,(mod\ J(R))$. Finally, $st \in J(R)$ and we conclude by Lam's theorem that R is strongly IC.

By using the (folklore) result the strong regularity of elements passes to corner rings, we derive directly from Theorem 2.3 that corner rings of E-solid rings are E-solid. Another proof goes as follows: let $e, f, g, h \in \text{idem}(R)$ such that $f, h \in eRe$ and $f \sim_r g \sim_\ell h$. As $ef = f$ and $fg = g$ then $eg = g$, and dually as $he = h$ and $gh = g$ then $ge = g$. Finally $g = ege \in eRe$. This proves that 2-chains pass to corner rings. Next example proves that the converse statement is false in general: locally E-solid rings need not be E-solid.

Example 2.4. Let $R = \begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix} \subseteq \mathcal{M}_2(\mathbb{Z})$. The ring R is non-unital, and non-zero idempotents of the ring $\mathcal{M}_2(\mathbb{Z})$ have rank 1 (in $\mathcal{M}_2(\mathbb{Z})$). Let $E \in \text{idem}(R)$ be a non-zero idempotent matrix. As $\mathbb{Z}$ is projective-free, then E is conjugate to $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{M}_2(\mathbb{Z})$, $E = UE_1U^{-1}$ for some invertible matrix U. Let $A \in ERE$. Then $A = UE_1U^{-1}AUE_1U^{-1}$ hence $U^{-1}AU = E_1U^{-1}AUE_1 = zE_1$ for some $z \in \mathbb{Z}$, and $A = UE_1zE_1E_1U^{-1} = zE$. It follows that $ERE \subseteq \mathbb{Z}E$ is a commutative ring, and any regular element of ERE is strongly regular. Thus R is locally E-solid.

To prove that R is not E-solid, we use the characterization (1) of Theorem 2.3. By the general theory of radicals over Morita contexts [22, Theorem 1]

$$J(R) = \begin{pmatrix} J(\mathbb{Z}) & * \\ * & J(2\mathbb{Z}) \end{pmatrix} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Consider the two matrices E_1 and $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Then $A, E_1 \in V(E_1)$ but $A - E_1 = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$ does not belong to $J(R)$. Thus reflexive inverses are not unique modulo the Jacobson radical and R is not E-solid.

Finally, by using arguments of [10], we derive the following result that deals with unitizations and square stable range 1.

Corollary 2.5. *Let R be a ring. Then the following statements are equivalent:*
(1) *R is E-solid;*
(2) *for any unitization $\hat{R}$ and any $a \in R, b \in \hat{R}$, if a is regular and $a\hat{R} + b\hat{R} = \hat{R}$ then $a^2 + by$ is a unit in $\hat{R}$ for some $y \in \hat{R}$;*
(3) *for some unitization $\hat{R}$ and any $a \in R, b \in \hat{R}$, if a is regular and $a\hat{R} + b\hat{R} = \hat{R}$ then $a^2 + by$ is a unit in $\hat{R}$ for some $y \in \hat{R}$.*

Proof. The implication (2) $\Rightarrow$ (3) is a tautology. To prove that (1) $\Rightarrow$ (2), assume that R is E-solid, and let $\hat{R}$ be any unitization of R. Let also $a \in R$ be regular. As R is E-solid then by Theorem 2.3 a is strongly regular in R hence in $\hat{R}$. By [10, Theorem 5.2] the element a has (right) square stable range 1 in $\hat{R}$, that is for any $b \in \hat{R}$ such that $a\hat{R} + b\hat{R} = \hat{R}$ then $a^2 + by$ is a unit in $\hat{R}$ for some $y \in \hat{R}$.
Finally, we prove that (3) $\Rightarrow$ (1). So assume (3) for some unitization $\hat{R}$. Let $a \in R$ be regular with reflexive inverse $a' \in R$. As $aa'\hat{R} = a\hat{R}$ with aa' idempotent then $a\hat{(R)}$ is Dedekind-finite (as a right R-module) iff the corner ring $C = aa'\hat{R}aa'$ is Dedekind-finite (as a unital ring). Let $x, y \in C$ be such that $xy = 1$. Then $yxy = y$ and y is regular. Moreover, $y \in R$ since $a \in R$ and R is an ideal of $\hat{R}$. By (4), as $y\hat{R} + (1-yx)\hat{R} = \hat{R}$ then $y^2 + (1-yx)z$ is a unit for some z. Thus $y = (y^2 + (1-yx)z)t$ for some $t \in \hat{R}$ and right multiplying by $x = xyx$ we obtain that $1 = xy = xyxy = xyxy^2t = y^2t$ since $yx(1-yx) = 0$. Finally y is a left and right invertible, $x = xy^2t = yt$ and $yx = 1$. This proves that $a\hat{R}$ is Dedekind-finite. By [10, Theorem 5.2], a is then strongly regular in $\hat{R}$ hence admits a commuting reflexive inverse $a^\sharp \in \hat{R}$. But $a^\sharp$ satisfies the equation $a^\sharp = (a^\sharp)^2 a$ hence lies in the ideal R of $\hat{R}$. This proves that R is strongly IC (equiv. E-solid). ☐

The last two equivalent conditions may be seen as a "regular square stable range 1" condition for non-unital rings in the spirit of Vaserstein [23].

Remarks
(1) As the Jacobson radical of regular rings is 0, we recover that R is E-solid regular iff R is abelian regular (equiv. Clifford, equiv. strongly regular, equiv. inverse, equiv. orthodox);
(2) If nilpotents of R belong $J(R)$ (equiv. are central modulo $J(R)$), then idempotents are central modulo $J(R)$ (note that for all $x \in R$ and $e \in \mathrm{idem}(R)$, $n = ex - exe$ is nilpotent);
(3) This happens also if $R/J(R)$ is abelian (because the quotient map sends idempotents of R to idempotents of $R/J(R)$). For instance, in [4, Theorem B] (see also [7, Proposition 4.5]), it is proved that any *adjoint regular ring* with a principal idempotent $e \in \mathrm{idem}(R)$

decomposes as a direct sum of subrings $R = eRe \oplus J(R)$, with eRe strongly regular. Consequently, such rings are E-solid.

(4) Recently, Lam [14] introduced *q-abelian rings* (where $eR(1-e)Re = 0$ for all $e \in \mathrm{idem}(R)$) and proved that they are strongly IC (E-solid). He also proved that $\mathcal{T}_2(S)$ is q-abelian iff S is abelian; Consequently, if S is E-solid but not abelian, then $\mathcal{T}_2(S)$ is E-solid (by [5, Theorem 3.8]) but not q-abelian.

(5) In [3], the authors consider the following example: S denotes the localization of $\mathbb{Z}$ at $3\mathbb{Z}$, and Q the quaternions over the ring S. They prove that $\mathrm{Nil}(Q) \subseteq J(Q)$, so that regular elements of Q are strongly regular (actually, as Q is a noncommutative domain, regular elements are either 0 or invertible), and that $T = Q/J(Q) \simeq \mathcal{M}_2(\mathbb{Z}_3)$. It follows that T contains non-central idempotents. This proves that for a ring R, the equivalent conditions of Theorem 2.3 are strictly weaker that $R/J(R)$ being abelian.

(6) On the other hand, the condition $\mathrm{Nil}(R) \subseteq J(R)$ (being *J-reduced*) is not necessary for R to be E-solid. Consider the (non-unital) semigroup algebra $R = \mathbb{F}_2\langle a, b | a^2 = b^2 = 0\rangle$. Then a is nilpotent but ab is not quasi-regular, and $a \notin J(R)$. However, R is trivially E-solid since its only idempotent is 0.

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