On the converse of a theorem of Harte and Mbekhta: Erratum to "On generalized inverses in C^*-algebras"

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Abstract. We prove that the converse of Theorem 9 in "On generalized inverses in C^*-algebras" by Harte and Mbekhta[2] is indeed true.

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In their article "On generalized inverses in C^*-algebras"[2], Harte and Mbekhta give the following theorem (A is a C^*-algebra):

Theorem 1. *A normalized commuting inverse is unique. If $a \in A$ has a commuting generalized inverse then it is decomposably regular, and*

$$A = aA + a^{-1}(0) \text{ with } aA \cap a^{-1}(0) = \{0\} \tag{1}$$

$$A = Aa + a_{-1}(0) \text{ with } Aa \cap a_{-1}(0) = \{0\} \tag{2}$$

and say "The conditions 1 and 2 are not together sufficient for $a \in aAa$ to be simply polar" (*i.e.* to have a commuting generalized inverse) and they exhibit a counterexample. The latter sentence is false for their conditions actually imply simple polarity of a:

Theorem 2. *Let A be a monoid (semigroup with identity) with involution. Then the following propositions are equivalent:*

1. *$a \in A$ is simply polar.*
2. *$Aa = Aa^2$ and $aA = a^2A$.*

Remark that the latter conditions are weaker than those in [2] (just multiply equation 1 left by a and equation 2 right by a), and that A needs not to be a ring.

Before giving the proof of the theorem, let us describe the original mistake of Harte and Mbekhta.

It is not true that both conditions (1) and (2) (conditions (9.1) and (9.2) in [2]) are satisfied by the example on page 75, lines 6 to 4 from the bottom, because if that were true then it would satisfy the relations $Aa = Aa^2$ and $aA = a^2A$, hence for the operator a there would exist an operator c such that $a = ca^2$.

Such an operator exists, namely $c = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, where $(Ax)_n = nx_n$, but it is not a bounded operator and hence it does not belong to the algebra of bounded operators on the square of the space c_{00}. As pointed out by the referee, on page 250 of the book [3] by Harte it is explicitly written that a certain operator is unbounded. It is exactly that operator whose boundedness is asserted falsely in the paper [2].

Proof of Theorem 2. Let $a^\sharp$ be the commuting inverse of a. Then $\forall c \in A$,

$$ca = caa^\sharp a = ca^\sharp aa$$

and $Aa = Aa^2$, and

$$ac = aa^\sharp ac = aaa^\sharp c$$

and $aA = a^2 A$.
Conversely, suppose that $Aa = Aa^2$ and $aA = a^2 A$. Then $\exists, b, c \in A^2$

$$a = aab = caa$$

It follows that

$$ab = ca^2 b = ca$$

and

$$\begin{aligned} a &= aab = aca \\ a &= caa = aba \end{aligned}$$

Define $e = cab$. Then e is an inner inverse:

$$aea = acaba = aba = a$$

and e is normalized:

$$eae = cabacab = cabab = cab = e$$

But also

$$ea = caba = ca = ab = acab = ae$$

and e commutes with a. Finally a is simply polar. □

This result can also be deduced from Green's relations (Theorem 7 in [1]).

References

1. J. A. Green, *On the structure of semigroups*, Ann. of Math. **54** (1951), no. 1, 163–172.
2. R. Harte and M. Mbekhta, *On generalized inverses in C*-algebras*, Studia Math. **103** (1992), 71–77.
3. R. E. Harte, *Invertibility and singularity*, Marcel Dekker, New York, 1988.

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