A local structure theorem for stable, $\mathcal{J}$-simple semigroup biacts

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Abstract

We describe a class of semigroup biacts that is analogous to the class of completely simple semigroups, and prove a structure theorem for those biacts that is analogous to the Rees-Sushkevitch Theorem. Precisely, we describe stable, $\mathcal{J}$-simple biacts in terms of wreath products, translations of completely simple semigroups, biacts over endomorphism monoids of free G-acts, tensor products and matrix biacts. Applications to coproducts and left acts are given.

Keywords Semigroup acts; Green's relations; stability; wreath products; endomorphism monoid of free G-acts; completely simple semigroups; Rees matrix semigroups

1 Introduction and notations

A large part of semigroup theory is devoted to their structure. Within this approach we distinguish (at least) two types of results. One is the so-called *local* theory, which provides fine descriptions of semigroups in certain classes - such as that of completely simple semigroups - and traces its origins back to Sushkevitch [28], Rees [23] and Clifford [6]. The second is the *global* theory, which to a large extent relies on the Krohn-Rhodes Theorem [13, 14, 24] and gives a broad description of the large class of finite semigroups. Historically, local theory came first, and makes great use of Green's relations on principal ideals [8]. The global theory, with its approach to decomposing semigroups using wreath products, and classifying up to divisibility, is more modern. It is interesting to note that the first proofs of the Krohn-Rhodes Theorem actually relied on the study of transformation monoids, that is, on monoid acts.

Whether or not the *global* theory of semigroups naturally applies to semigroup acts, there has been (to the knowledge of the author) only a few attempts to develop directly a *local* theory of semigroup acts, at least not without strong assumptions on the semigroup (as for instance in [3], [17] or [19], where the semigroups are assumed to be left or right zero semigroups, or completely 0-simple semigroups, see also [15] and [21] for actions of inverse semigroups). One instance of a general approach is [20], and the other is Section 3 of [27]. Oehmke's work [20] relies on a condition slightly weaker than finiteness, and the study of certain congruences on S induced by elements of the S-act. Steinberg [27] relies on a proper study of certain (finite) transformation semigroups. Neither of these works is based explicitly on notions corresponding to Green's relations, even if both need the action to be transitive (thus $\mathcal{L}$-simple in the sense we introduce later). In particular both Oehmke and Steinberg consider cyclic acts only. This may be due to the lack of symmetry of (one-sided) semigroup acts. Our aim is to show that a general (local) theory of semigroup biacts can be constructed, similar, but not equivalent to, the local theory of semigroups, by using Green's relations on these semigroup biacts. We emphasise that we do this without any assumption on the semigroup. Precisely, we prove in Section 4 that faithful, stable, $\mathcal{J}$-simple biacts (defined in Sections 2 and 3) admit descriptions up to isomorphism (and not just division) in terms of:

1. wreath products involving a group G;
2. left and right translations over a completely simple semigroup (with structure group G);
3. biacts over endomorphism monoids of free G-acts, where G is a group;
4. tensor products of biacts over a group G;
5. matrix biacts with coefficients in a group G.

This theory may then be fruitfully applied to one-sided semigroup acts (Section 5).

While reading the present article, it is important to have in mind the following principle that served as a guideline for this research: in order to properly understand a general semigroup biact, we believe that one must forget about the semigroups *per se* and consider only the action of the elements of the semigroups on the elements of the act. The reader may think of this as an automaton-theoretic point of view. In particular, this means that we use only recursively enumerable sets or relations (formulas with the existential quantifier only) on the level of elements.

We will use the following conventions for functions on a set. For any two sets A and B, B^A denotes the set of functions from A to B. If X is a set, by $\mathcal{T}(X)$ we mean the full transformation monoid on X, with composition as binary operation, where we write f on the left of its argument $x \in X$. That is $\mathcal{T}(X) = (X^X, \circ)$ with $f \circ g : x \mapsto f(g(x))$. By $\mathcal{T}^{op}(X)$ we mean the opposite monoid $(X^X, \circ^{op})$ with $f \circ^{op} g : x \mapsto g(f(x))$, where we write functions on the right of their arguments and denote $\circ^{op}$ by juxtaposition, so that $f \circ^{op} g = fg : x \mapsto x(fg) = (xf)g$.

To avoid pathological cases, all our semigroups are non-empty. If S is a semigroup (or a monoid, or a group), we will denote by $\underline{S}$ its underlying set to emphasize its role as a set (and by constrast, the role of S as a semigroup) as in the following sentence: "Let S be a semigroup. Then $\underline{S}$ is a right S-act with action $x \odot s = xs, x \in \underline{S}, s \in S$." We denote the fact that S is a subsemigroup of T by $S \trianglelefteq T$.

Finally, we try as much as possible to use the following conventions. Lowercase Latin letters will denote elements, or functions on a set. Lowercase Greek letters will denote functions with additional structure (morphisms between semigroups, semigroup actions...), or congruences. Uppercase letters will denote sets. An exception to these conventions is made for Green's relations, in order to fit with the traditional notation. We thus use script uppercase letters for Green's relations and Latin uppercase letters for the associated classes.

2 Semigroup acts and biacts, categories

2.1 Semigroup acts and biacts, (T, S)-biacts

A *right semigroup act* is a triple $\mathbf{X} = (X, S, \beta)$ where X is a set, S is a semigroup, and $\beta : X \times S \to X$ is a *semigroup action*, that is, a function such that for all $s, s' \in S$ and $x \in X$ $\beta(x, ss') = \beta(\beta(x, s), s')$. A *right monoid act* is a right semigroup act $\mathbf{X} = (X, N, \beta)$ where N is a monoid and β is a unitary semigroup action, that is, in addition to being a semigroup action, we have $\beta(x, 1) = x$ for all $x \in X$.

By contrast, if S (resp. N) is a given semigroup (resp. monoid), then a *right S-act* (resp. *right N-act*) is just a pair (X, β) where $\beta : X \times S \to X$ (resp. $\beta : X \times N \to X$) is a semigroup (resp. monoid) action, and we simply say that X is a right S-act (resp. right N-act). If we want to emphasize the distinction between the right S-act and its underlying set, we may denote the former by X_S. Left semigroup or monoid acts are defined dually, as are left T-acts (resp. left M-acts) for T (resp. M) a given semigroup (resp. monoid).

We remark that S-acts appear under different names in the literature, including S-sets, S-systems, S-operands, S-polygons, S-automata, S-semimodules, monars etc.

By a *semigroup biact*, we mean a 5-tuple $\mathbf{X} = (T, X, S, \alpha, \beta)$ where (T, X, α) and (X, S, β) are left and right semigroup acts and the following compatiblity condition holds:

$$(\forall t \in T, \forall x \in X, \forall s \in S)\ \alpha(t, \beta(x, s)) = \beta(\alpha(t, x), s).$$

A *monoid biact* $\mathbf{X} = (M, X, N, \alpha, \beta)$ is defined accordingly. When T and S are fixed, the triple (X, α, β) (usually abbreviated to X or ${}_T X_S$) is a (T, S)-biact. For any $t \in T, x \in X$ and $s \in S$, when no confusion is possible, we will simply denote $\alpha(t, x)$ by tx (or $t \cdot x$) and $\beta(x, s)$ by xs (or $x \odot s$) and simply refer to the biact as the triple $\mathbf{X} = (T, X, S)$. The compatiblity condition then reads

$$(\forall t \in T, \forall x \in X, \forall s \in S)\ t(xs) = (tx)s$$

and the expression $txs = t(xs) = (tx)s$ is unambiguous.

In this article, we prefer to work with semigroup biacts than semigroup acts for reasons of symmetry and duality, that will become more obvious in the study of *Green's relations* on biacts.

An important example of a monoid biact is the biact of matrices over a given ring. This example will lead to a convenient representation of certain biacts in the sequel.

Example 2.1. Let R be a ring. For any positive integers p, q denote by $M_{p,q}(R)$ the set of matrices with p rows and q columns. Now fix p and q. Endow $M_{p,p}(R)$ and $M_{q,q}(R)$ with the matrix product. Then

$$\boldsymbol{R_{p,q}} = (M_{p,p}(R), M_{p,q}(R), M_{q,q}(R))$$

with biaction the matrix product is a monoid biact.

Actually, the previous biact may be derived from the following construction:

Example 2.2. Let S be a semigroup and e, f two idempotents of S. Then $(eSe, e\underline{S}f, fSf)$ is a monoid biact.

To recover Example 2.1, set $S = M_{p+q,p+q}(R)$, $e = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 0 & I_q \end{pmatrix}$ and identify $M_{p,q}(R)$ with the upper right corner $e\underline{S}f = \begin{pmatrix} 0 & M_{p,q}(R) \\ 0 & 0 \end{pmatrix}$.

2.2 The category $(T, S)-$Biact

Let S and T be semigroups. We form the category **$(T, S)-$Biact** as follows:

- Objects are (T, S)-biacts (X, α, β) (thus, α and β are compatible actions);
- Morphisms $(X, \alpha, \beta) \to (X', \alpha', \beta')$ are functions $f : X \to X'$ compatible with the semigroup actions, that is,

$$(\forall t \in T, \forall s \in S, \forall x \in X)\ f(tx) = tf(x),\ f(xs) = f(x)s.$$

The categories **Left T-act** and **Right S-act** are defined accordingly. Products and coproducts exists in those three categories. They are defined as the classical product and coproduct (disjoint union) of sets with the obvious induced actions. A direct proof of the coproduct case is given in [11] Proposition 2.1.8. A direct proof for products can be deduced from the fact that the category **Left T-act** is actually an equational category (Example 13.15 in [1]). Thus by [1, Proposition 1.21] products (indeed, more generally, limits) exist and are built up at the level of sets.

By a (T, S)-*subact* of the (T, S)-biact (X, α, β) we mean a subset $Y \subseteq X$ such that $TY \cup YS \cup TYS \subseteq Y$, endowed with the restrictions of the actions.

2.3 The category SemBiact

It will also be useful to consider the larger category of biacts over any semigroups. We therefore form the category **SemBiact** as follows:

- Objects are semigroups biacts $\mathbf{X} = (T, X, S)$;
- Morphisms $(T, X, S) \to (T', X', S')$ are triples $\Phi = (\phi, f, \psi)$ where $f : X \to X'$ is a function, $\phi : T \to T'$ and $\psi : S \to S^{'}$ are semigroup morphisms and

$$(\forall t \in T, \forall s \in S, \forall x \in X)\ f(tx) = \phi(t)f(x),\ f(xs) = f(x)\psi(s).$$

The categories **LeftSemAct** and **RightSemAct** are defined accordingly.

A morphism $\Phi = (\phi, f, \psi) \in Hom(\mathbf{X}, \mathbf{X}')$ is epi if ϕ, f and ψ are onto. It is mono if ϕ, f and ψ are one-to-one. It is an isomorphism if ϕ, f and ψ are bijections; note that in this case the reciprocals automatically define morphisms.

Example 2.3. (The free semigroup biact over $\{t\}, \{x\}, \{s\}$.)
Let $\{t\}, \{x\}, \{s\}$ be three one-element sets. Let $T = \langle t \rangle$ be the free semigroup generated by t, and $S = \langle s \rangle$ be the free semigroup generated by s. Finally define $X = \{t^k x s^l, k, l \geq 0\}$ (with the convention $t^0 x = x = x s^0$) with actions for any $p, q > 0$, $k, l \geq 0$, $t^p \cdot (t^k x s^l) = t^{p+k} x s^l$ and $(t^k x s^l) \odot s^q = t^k x s^{l+q}$. Then $\mathbf{X} = (T, X, S)$ is the free semigroup biact over $\{t\}, \{x\}, \{s\}$.

We will also use the notion of tensor product of biacts (see [11]). Let $\mathbf{X} = (T, X, R)$ and $\mathbf{Y} = (R, Y, S)$ be two semigroup biacts. Define the equivalence relation ν on $X \times Y$ generated by all pairs of the form $((xr, y), (x, ry))$. Then $X \otimes Y = X \times Y / \nu$ is called the *tensor product* of the right act (X, R) and the left act (R, Y) (and indeed satisfies a universal property). The semigroups T and S acts on $X \otimes Y$ in a compatible way so that the biact $\boldsymbol{X} \otimes \mathbf{Y} = (T, X \otimes Y, S)$ is well defined.

2.4 The regular representation

Let X be a set and T be a subsemigroup of $\mathcal{T}(X)$. Then T acts on X on the left by evaluation, so that any transformation semigroup defines a left semigroup act. Conversely, a classical argument associates to any left T-act ${}_T X$ a subsemigroup of $\mathcal{T}(X)$ that is a homomorphic image of T. The morphism $\phi : T \to \mathcal{T}(X)$ is defined by $\phi(t) = \delta_t$ for all $t \in T$, where $\delta_t : x \mapsto tx$ is the left translation on X induced by t. This construction extends to biacts as follows.

Two subsemigroups $T \trianglelefteq \mathcal{T}(X)$ and $S \trianglelefteq \mathcal{T}^{op}(X)$ are *compatible* if they commute as functions from X to X, that is for any $x \in X, f \in T$ and $g \in S$ we have that $(f(x))g = f(x)g$. If this is the case, they define a semigroup biact (T, X, S).

Conversely, let $\mathbf{X} = (T, X, S)$ be an object in **SemBiact**. For any $t \in T$ one can define the left translation $\delta_t \in \mathcal{T}(X)$ by $\delta_t : x \mapsto tx$ (resp. for any $s \in S$, the right translation $\tau_s \in \mathcal{T}^{op}(X)$ by $\tau_s : x \mapsto xs$). Then $\phi : T \to \mathcal{T}(X),\ t \mapsto \delta_t$ is a semigroup homomorphism from T to $\mathcal{T}(X)$ and dually, $\psi : S \to \mathcal{T}^{op}(X),\ s \mapsto \tau_s$ is a semigroup homomorphism from S to $\mathcal{T}^{op}(X)$, such that $\phi(T)$ and $\psi(S)$ are compatible. Putting $RegT = \phi(T)$ and $Reg(S) = \psi(S)$ we have the biact $\mathbf{RegX} = (RegT, X, RegS)$ is the *regular representation* of $\mathbf{X} = (T, X, S)$.

If $\Phi = (\phi, id_X, \psi)$ is an isomorphism then we say that T and S act faithfully on X, or that $\mathbf{X} = (T, X, S)$ is a faithful biact.

Example 2.4. Let (T, X, S) be a semigroup biact with X finite. Then for any $t \in T$ there exits a power $t^k \in T, k \geq 1$ whose action on X is idempotent. Indeed, consider the regular representation $(RegT, X, RegS)$. As X is finite then $\mathcal{T}(X)$ (hence $RegT$) are finite semigroups and $\delta_t \in RegT$ has an idempotent power $\delta_t^k = \delta_{t^k}$. For any $x \in X$, it then holds that $t^k t^k x = \delta_{t^k}\delta_{t^k} x = \delta_{t^k} x = t^k x$.

This representation by functions is very close to the classical case, but it can in certain cases be interestingly replaced by the following one.

Let (T, X, S) be a semigroup biact. Then ${}_TX$ is a left T-act and it makes sense to define *T-endomorphisms* as elements of the *endomorphism monoid* $End^{op}\left((T, X)\right) = End^{op}({}_TX) = Hom^{op}({}_TX, {}_TX)$ in the category **Left *T*-act** (the "op" meaning that product is conjugation in reverse order). Dually, we can also define $End\left((X, S)\right) = End(X_S) = Hom(X_S, X_S)$ in the category **Right *S*-act**. As $(tx)s = t(xs)$ for all $t \in T, s \in S, x \in X$ then the right translation τ_s actually defines an element of $End^{op}({}_TX)$, and the left translation δ_t actually defines an element of $End(X_S)$.

In the sequel, we will therefore mostly consider $RegT$ as a submonoid of $End(X_S)$ and $RegS$ as a submonoid of $End^{op}({}_TX)$ rather as submonoids of functions.

In particular, we will use the following construction (inspired by the construction of the dual in functional analysis): Let (T, X) be a left semigroup act (equivalently, let X be a left T-act). Then $End^{op}({}_TX)$ is a monoid, that acts on X on the right by point evaluation: $x \odot g = [x]g\ x \in X, g \in End^{op}({}_TX)$, such that $(T, X, End^{op}({}_TX))$ is a semigroup biact. The dual construction holds.

Lemma 2.5. *Let (T, X, S) be a faithful biact. Then (T, X) embeds in the left act $(End(X_S), X)$, and dually.*

Proof. As the biact is faithful, then $T \sim RegT \trianglelefteq End(X_S)$. □

Example 2.6. Consider the right act $(X, 1)$. Then any function from X to X is an 1-endomorphism and $(End(X_1), X) = (\mathcal{T}(X), X)$.

Example 2.7 (See also [27])**.** Let M be a monoid, e an idempotent of M. Then M acts on the principal left ideal $\underline{M}e$ on the left, so that $(M, \underline{M}e)$ is a left monoid act. It holds that $(\underline{M}e, End^{op}({}_M\underline{M}e)) \sim (\underline{M}e, eMe)$ with right multiplication as right action. First, the map $\tau : eme \mapsto \tau_{eme}$ is injective because $\underline{M}e \ni e$. Second, as multiplication in a monoid is associative, then $\tau(eMe) \trianglelefteq End^{op}({}_M\underline{M}e)$. Finally, let $g \in End^{op}({}_M\underline{M}e)$ and pose $m' = e[e]g$. By construction $m' \in eMe$. Let $x = me = me^3 \in \underline{M}e$. Then $[x]g = [me^3]g = mee[e]g = mem' = xm'$ and $\tau_{m'} = g$. This ends the identification.

It may happen that the endomorphism monoid $End^{op}({}_TX)$ is too large for our purpose, and we will sometimes replace it with $Aut^{op}({}_TX)$, automorphism monoid of ${}_TX$.

3 Analysis of monoid biacts by Green's relations

In this section, we consider a given semigroup biact $\mathbf{X} = (T, X, S)$, or equivalently a fixed (T, S)-biact ${}_TX_S$.

3.1 Green's relations and Green's lemma for semigroup biacts

Analogously to Green's relations, we define the following relations on the (T, S)-biact ${}_TX_S$. These relations are defined in [11], but only few results (see Lemma 3.3) are derived from these definitions. As usual S^1 (resp. T^1) denotes the monoid generated by S (resp. T). Let $x, y \in X$.

1. $x \,\mathcal{R}\, y \Leftrightarrow (\exists s, s' \in S)\, xs = y$ and $ys' = x \Leftrightarrow xS^1 = yS^1$.
2. $x \,\mathcal{L}\, y \Leftrightarrow (\exists t, t' \in T)\, tx = y$ and $t'y = x \Leftrightarrow T^1x = T^1y$.
3. $\mathcal{H} = \mathcal{R} \wedge \mathcal{L}$.
4. $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$.
5. $x \,\mathcal{J}\, y \Leftrightarrow (\exists t, t' \in T, \exists s, s' \in S)\, txs = y$ and $t'ys' = x \Leftrightarrow T^1xS^1 = T^1yS^1$.

It follows from their definition that these relations are equivalence relations on X. Relation $\mathcal{R}$ (resp. $\mathcal{L}$) is a left (resp. right) congruence, that is for any $t \in T$, $x \,\mathcal{R}\, y$ implies $tx \,\mathcal{R}\, ty$. We will frequently use the following cancellation property, well known for semigroups: for any $x \,\mathcal{R}\, y$ and $t, t' \in T$, if $tx = t'x$ then $ty = t'y$, and dually for $\mathcal{L}$.

Let $\mathcal{K}$ denote any of these relations. Then the semigroup biact $\mathbf{X} = (T, X, S)$ (resp. the (T, S)-biact ${}_TX_S$) is $\mathcal{K}$-simple if X consists of a single $\mathcal{K}$-class. Thus,the statement "the (left) action of T on X is transitive" is equivalent to "the left T-act ${}_TX$ is $\mathcal{L}$-simple".

Example 3.1. let H and K be subgroups of a group G, and consider the biact $(H, \underline{G}, K)$ with multiplication as actions. Let $x, y \in \underline{G}$. Then $x \,\mathcal{L}\, y \Leftrightarrow Hx = Hy \Leftrightarrow x \in Hy$ since H is a group, and $\mathcal{L}$-classes are left cosets. Dually, $\mathcal{R}$-classes are right cosets and $\mathcal{J}$-classes are two sided cosets $J_x = HxK$. The $\mathcal{H}$-class of 1 is $H \cap K$.

We will need the following (very straightforward) lemma:

Lemma 3.2. *Let (T, X) be a $\mathcal{L}$-simple left semigroup act, and $x \in X$. Then $T^1x = Tx = X$.*

Proof. As (T, X) is $\mathcal{L}$-simple, then $T^1x = T^1y$ for any $y \in X$ and $X \subseteq T^1x \subseteq X$. Let now $t \in T$. As $y = tx \, \mathcal{L} \, x$ then exists $t' \in T^1, t'tx = x$ with $t't \in T$ so that $x \in Tx$ and $T^1x = Tx$. □

In the rest of the section, we show that most of the classical results in semigroup theory regarding Green's relations admit a biact version.

Lemma 3.3. *It holds that* $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ *and* $\mathcal{D} \subseteq \mathcal{J}$*.*

This is Proposition 4.48 in [11]. We recall the proof for self-containedness.

Proof. Let $x, y, z \in X$ such that $x \, \mathcal{R} \, y \, \mathcal{L} \, z$. Then exists $s, s' \in S^1$, $x = ys$ and $y = xs'$, and exists $t, t' \in T^1$ such that $ty = z$ and $t'z = y$. It follows that $y = t'ty = yss'$ and as $x \, \mathcal{R} \, y$ and $y \, \mathcal{L} \, z$, then $x = t'tx$ and$z = zss'$. Pose $u = tys = tx = zs$. Then $z \, \mathcal{R} \, u \, \mathcal{L} \, x$ as required. By duality $\mathcal{R}$ and $\mathcal{L}$ commute, and $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$.
The inclusion $\mathcal{D} \subseteq \mathcal{J}$ then follows from $\mathcal{R}, \mathcal{L} \subseteq \mathcal{J}$. □

Example 3.4 (Bicyclic biact)**.** Let $T =< t >$ be the free semigroup generated by t, and $S =< s >$ be the free monoid generated by s. Let $\{x\}$ be a one element set and pose $X = \{x = t^0x = xs^0, t^px, xs^q, p, q > 0\}$. There are compatible semigroup actions defined for any $p, q > 0, k \geq 0$ by $t^p \cdot (t^kx) = t^{p+k}x$, $t^p \cdot (xs^q) = t^{p-q}x = (t^px) \odot s^q$ if $p - q \geq 0$ and $t^p \cdot (xs^q) = xs^{q-p} = (t^px) \odot s^q$ if $p - q \leq 0$ and $(xs^k)s^q = xs^{k+q}$. The semigroup biact $\mathbf{X} = (T, X, S)$ is the quotient of the free semigroup biact over $\{t\}, \{x\}, \{s\}$ by the relation generated by $txs = x$. By analogy with the semigroup case, we call this biact the bicyclic biact. In $\mathbf{X}$, it holds that $\mathcal{L} = \mathcal{R} = \mathcal{D} = \Delta$ the diagonal of $X \times X$. But relation $\mathcal{J}$ is the universal relation as $TyS = X$ for any $y \in X$.

Green's lemma holds for these relations.

Lemma 3.5. *Let* $x, y \in X$ *and* $s, s' \in S^1$ *such that* $xs = y$ *and* $ys' = x$ $(x \, \mathcal{R} \, y)$*. Then the right translation* $\tau_s : z \mapsto zs$ *is a bijection from* L_x *to* L_y *with inverse* $\tau_{s'}$*, that preserves* $\mathcal{R}$*-classes. In particular it sends* $\mathcal{H}$*-classes to* $\mathcal{H}$*-classes.*

Proof. Let $x, y \in X$ and $s, s' \in S^1$ as in the lemma and let $z \in L_x$. Then by right congruence $zs \, \mathcal{L} \, xs = y$ and τ_s maps L_x to L_y. Symmetrically $\tau_{s'}$ maps L_y to L_x. Also by cancellation, as $xss' = x$ then $zss' = z$ and as $ys's = y$ then $zs's = z$, and τ_s, $\tau_{s'}$ are reciprocal (also, as $zss' = z$ then $zs \, \mathcal{R} \, z$). □

By duality:

Lemma 3.6. *Let* $x, y \in X$ *and* $t, t' \in T^1$ *such that* $tx = y$ *and* $t'y = x$ $(x \, \mathcal{L} \, y)$*. Then the left translation* $\delta_t : z \mapsto tz$ *is a bijection from* R_x *to* R_y *with inverse* $\delta_{t'}$ *that preserves* $\mathcal{L}$*-classes. In particular it sends* $\mathcal{H}$*-classes to* $\mathcal{H}$*-classes.*

We deduce directly from these lemmas that within a single $\mathcal{D}$-class, the $\mathcal{H}$-classes are in bijection.

Corollary 3.7. *Any two* $\mathcal{H}$*-classes in the same* $\mathcal{D}$*-class are equipotent.*

As a second consequence of Lemmas 3.5 and 3.6 we get:

Corollary 3.8. *Let $x, y \in X$, $t, t' \in T^1$ and $s, s' \in S^1$. Then*

1. *Either $H_x s \cap R_x = \emptyset$ or $H_x s \cap R_x = H_{xs} = H_x s$;*
2. *If $H_x s \cap R_x = \emptyset$ then $H_x ss' \cap R_x = \emptyset$;*
3. *If $H_x s \cap R_x = \emptyset$ and $tx\mathcal{L}x$ then $H_{tx} s \cap R_{tx} = \emptyset$;*
4. *Either $tH_x \cap L_x = \emptyset$ or $tH_x \cap L_x = H_{tx} = tH_x$;*
5. *If $tH_x \cap L_x = \emptyset$ then $t'tH_x \cap L_x = \emptyset$;*
6. *If $tH_x \cap L_x = \emptyset$ and $xs\mathcal{R}x$ then $tH_{xs} \cap L_{xs} = \emptyset$.*

Proof. We prove only the three first statements. The others are dual.

1. Assume $H_x s \cap R_x$ contains an element $y = zs$ with $z \,\mathcal{H}\, x$. By Lemma 3.5 right multiplication by s is a bijection from L_x to L_y that sends $\mathcal{H}$-classes to $\mathcal{H}$-classes, so that $H_y = H_x s$. But $H_y = H_y \cap R_y$ and as $y \,\mathcal{R}\, x$, the conclusion follows.

2. Assume $H_x ss' \cap R_x$ contains an element $y = zss'$ with $z \,\mathcal{H}\, x$. Let us prove that $zs \,\mathcal{R}\, x$. As $y \,\mathcal{R}\, x$ exists $u, u' \in S^1$, $xu = y$ and $yu' = x$. Thus $zss'u' = yu' = x$. Also as $z \,\mathcal{R}\, x$ exists $v, v' \in S^1$ such that $xv = z$ and $zv' = x$, and $zs = xvs$. Finally $zs \,\mathcal{R}\, x$ and $H_x s \cap R_x$ is not empty.

3. Assume $tx \,\mathcal{L}\, x$ and $H_{tx} s \cap R_{tx}$ is not empty. Then by the first item $H_{tx} s \cap R_{tx} = H_{txs} = H_{tx} s$ and $txs \,\mathcal{R}\, tx$, so that exists $s' \in S^1$ such that $txss' = tx$. Also, as $tx \,\mathcal{L}\, x$ then exists $t' \in T^1$ such that $t'tx = x$. Finally $x = t'tx = t'txss' = xs'$ and $xs \in H_x s \cap R_x$.

□

In particular, we have the equivalences $H_x s \subseteq H_x \Leftrightarrow H_x s = H_x \Leftrightarrow xs\mathcal{H}s$ and their dual $tH_x \subseteq H_x \Leftrightarrow tH_x = H_x \Leftrightarrow tx \,\mathcal{H}\, x$. Also, we have the following corollary:

Corollary 3.9. *Let $x, y \in X$, $t \in T^1$ and $s \in S^1$.*

1. *If $x \,\mathcal{R}\, y$, then $tx \,\mathcal{H}\, x \Leftrightarrow ty \,\mathcal{H}\, y$.*
2. *If $x \,\mathcal{L}\, y$, then $xs \,\mathcal{H}\, x \Leftrightarrow ys \,\mathcal{H}\, y$.*

Proof. Assume that $x \,\mathcal{R}\, y$ and $tx \,\mathcal{H}\, x$. By Lemma 3.6 left translation by t is a bijection from R_x to R_x that preserves $\mathcal{L}$-classes, so that $ty \in R_x \cap L_y = H_y$. This ends the proof. The other statement is dual. □

3.2 Stability

Definition 3.10. We say that the semigroup biact $\mathbf{X} = (T, X, S)$ (resp. the (T, S)-biact ${}_TX_S$) is *left stable* (resp. *right stable, stable*) if $x\,\mathcal{J}\,tx \Leftrightarrow x\,\mathcal{L}\,tx$ for any $x \in X, t \in T$ (resp. $x\,\mathcal{J}\,xs \Leftrightarrow x\,\mathcal{R}\,xs$ for any $x \in X, s \in S$, resp. both).
It is *completely left stable* (resp. *completely right stable, completely stable*) if $x\,\mathcal{L}\,tx$ for any $x \in X, t \in T^1$ (resp. $x\,\mathcal{R}\,xs$ for any $x \in X, s \in S^1$, resp. both).

The definition of left (resp. right) complete stability carries on to one-sided acts and is equivalent to $x\,\mathcal{L}\,tx$ for any $x \in X, t \in T$ (resp. $x\,\mathcal{R}\,xs$ for any $x \in X, s \in S$).

Example 3.11. Let S be a semigroup. then $(S, \underline{S}, S)$ is completely stable if and only if S is completely simple. Indeed, if $(S, \underline{S}, S)$ is completely stable, then for any $s, t \in S$, $s\,\mathcal{R}\,st\,\mathcal{L}\,t$ and S is $\mathcal{D}$-simple. Also, S is completely regular since for any $s \in S$, $s\,\mathcal{R}\,s^2\,\mathcal{L}\,s$, and thus s is group invertible by [8] Theorem 7 and corollary thereto. It follows that S is completely simple as a completely regular and $\mathcal{D}$-simple semigroup.
Conversely, if S is completely simple it is completely regular and $\mathcal{J}$-simple, and $\mathcal{D} = \mathcal{J}$. Let $t \in S, x \in S$. As $R_x \cap L_t$ is non void ($x\,\mathcal{D}\,t$) and the semigroup is completely regular then $R_x \cap L_t$ contains an idempotent element. By [18] Theorem 3, $tx \in L_x \cap R_t$ and the biact is left completely stable. We conclude by duality.

It holds that:

Lemma 3.12. *Let ${}_TX$ be a left T-act and $x \in X$. Then $tx\mathcal{L}x$ for all $t \in T$ if and only if $L_x = Tx$. In particular, a left T-act ${}_TX$ is completely left stable if and only if it is the coproduct (in* ***Left T-acts****) of its $\mathcal{L}$-classes, which are minimal cyclic T-subacts.*

Proof. First, for any $x \in X$, $L_x \subseteq T^1x$. And $tx\,\mathcal{L}\,x$ for all $t \in T$ if and only if $Tx \subseteq L_x$ is a tautology. We thus only have to prove that $x \in Tx$. Let $t \in T$ (T is non-void by assumption). Then $tx\,\mathcal{L}\,x$ and there exists $t' \in T^1, t'tx = x$. But $t't \in T$ so that $x \in Tx$ and finally $L_x = Tx$.
Thus, if X is completely left stable, it is the coproduct of its $\mathcal{L}$-classes, which are cyclic left subacts. Let $L = L_x$ be an $\mathcal{L}$-class, and $y \in L'$ subact of L. Then $y\,\mathcal{L}\,x$ and $Ty = L_y = L_x = L$ so that $L \subseteq L' \subseteq L$, and L is minimal. Conversely, assume that X is the coproduct of its $\mathcal{L}$-classes and let $x \in X$. Then L_x is a T-subact and $tx\,\mathcal{L}\,x$ for all $t \in T$. □

A prototypical example of completely left stable act is a space X with a left group action, for if G is a group acting on X, then for any $x \in X$ and $g \in G$ $g^{-1}.(g.x) = x$.

Example 3.13. Let G be a group acting on X on the left. Then ${}_GX$ is completely left stable, hence the coproduct of its orbits which are transitive spaces.

It is well known that finite semigroups are stable. So are finite biacts.

Lemma 3.14. *Any finite biact is stable.*

Proof. Let (T, X, S) be a biact with X finite, and $x \in X, s \in S$ such that $x \mathcal{J} xs$. Then there exist $u \in S^1, v \in S^1$ such that $vxsu = x$. It follows that for any $k \in \mathbb{N}$, $v^k x(su)^k = x$. Consider $\tau_{su} \in \mathcal{T}^{op}(X)$. As X is finite $\mathcal{T}^{op}(X)$ is a finite semigroup and there exists an integer $k \geq 1$ such that τ_{su}^k is idempotent. Then $xsu(su)^{k-1} = x(su)^k = v^k x(su)^k(su)^k = v^k x\tau_{su}^k\tau_{su}^k = v^k x\tau_{su}^k = v^k x(su)^k = x$. We conclude by duality. □

Lemma 3.15. *Let (T, X, S) be stable. Then $\mathcal{J} = \mathcal{D}$.*

Proof. We already know that $\mathcal{D} \subseteq \mathcal{J}$. We prove that $\mathcal{J} \subseteq \mathcal{D}$. Let $x, y \in X$ such that $x \mathcal{J} y$. Then there exist $s \in S^1, t \in T^1$ such that $x = tys$. Then $T^1yS^1 = T^1xS^1 = T^1tysS^1 \subseteq T^1tyS^1$, and also $T^1tyS^1 \subseteq T^1yS^1$. Hence $T^1tyS^1 = T^1yS^1$ and dually $T^1ysS^1 = T^1yS^1$. Hence $ty \mathcal{J} y \mathcal{J} ys$ and as (T, X, S) is stable, then $ty \mathcal{L} y \mathcal{R} ys$. As $\mathcal{L}$ is a right congruence then $tys \mathcal{L} ys$ and finally, $x \mathcal{L} ys \mathcal{R} y$, and $x \mathcal{D} y$ since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. □

Example 3.16. Let (T, X) be a left semigroup act, and pose $S = Aut^{op}({}_TX)$. Then (T, X, S) is completely right stable. Indeed, for any $x \in X, s \in Aut^{op}(({}_TX)$ it holds that $[x]ss^{-1} = x$.

Regarding completely stable biacts, we have:

Lemma 3.17. *Let ${}_TX_S$ be a completely stable (T, S)-biact. Then it is the coproduct of its $\mathcal{J}$-classes.*
Conversely, a stable (T, S)-biact that is the coproduct of its $\mathcal{J}$-classes is completely stable.

Proof. Assume that ${}_TX_S$ is completely stable and let $x \in X$, $t \in T^1, s \in S^1$. Then $txs \mathcal{L} xs \mathcal{R} x$ by complete stability and $txs \mathcal{D} x$, whence $T^1xS^1 \subseteq J_x$ ($\mathcal{D} \subseteq \mathcal{J}$ in general, and $\mathcal{D} = \mathcal{J}$ in our particular case). But $J_x \subseteq TxS$ since $J_x \subseteq T^1xS^1$ (by definition of $\mathcal{J}$) and $x \in TxS$ by complete stability and T, S being non empty semigroups (by convention), see Lemma 3.17. Thus the two sets are equal. In particular J_x is a (cyclic) (T, S)-subact, and since $\mathcal{J}$-classes form a partition of X, ${}_TX_S$ is the coproduct of its $\mathcal{J}$-classes.
Conversely, if ${}_TX_S$ is stable and the coproduct of its $\mathcal{J}$-classes, then each $\mathcal{J}$-class is a (T, S)-subact. If $t \in T, x \in X$ then $tx \mathcal{J} x$ and by stability, $tx \mathcal{L} x$. We conclude by duality. □

Corollary 3.18. *Let X_N be a unitary right N-act, with N a monoid. It is completely right stable and indecomposable if and only if it is of the form $X \sim N/\rho$, where ρ is a transitive right congruence on N: $(\forall m, m' \in N, \exists n \in N)\ mn\rho m'$.*

Proof. Consider the monoid biact $\mathbf{X} = (1, X, N)$ where 1 is the monoid with one element with unitary action on X. Assume first that X is completely right stable and indecomposable. Then $\mathbf{X}$ is completely stable hence the coproduct of its $\mathcal{J}$-classes, and since it is indecomposable $X = J_x = xN$ for some $x \in X$. As a cyclic right monoid act, it is of the form $X \sim N/\rho$, where ρ is a right

congruence (define ρ by $m\rho n \Leftrightarrow x_0m = x_0n$, where $x_0 \in X$ satisfies $X = x_0N$). And any two ρ-classes are $\mathcal{J}$ hence $\mathcal{R}$-related so that ρ is transitive. Indeed, let $m, m' \in N$. Then $x_0m \,\mathcal{R}\, x_0m'$ and there exists $n \in N$ such that $x_0mn = x_0m'$, and finally $mn\rho m'$.
Conversely, if $X \sim N/\rho$ for ρ a transitive right congruence on N, then $X = x_0N$ for some $x_0 \in X$ ($x_0 = \rho_1$ works) and any two elements of X are $\mathcal{R}$-related, whence it is completely right stable and indecomposable. □

Theorem 2.54 of O. Andersen [2] states that any simple semigroup which is not completely simple contains a copy of the bicyclic semigroup. We prove an analogous result for semigroup biacts.

Theorem 3.19. *Let $\mathbf{X} = (T, X, S)$ be a $\mathcal{J}$-simple semigroup biact and assume that there exist $x \in X, t \in T, s \in S$ such that $(tx, x) \notin \mathcal{L}$ and $(x, xs) \notin \mathcal{R}$. Then it contains a copy of the bicyclic biact.*

Proof. Let x, t, s as above. As $txs \,\mathcal{J}\, x$ then there exist $t' \in T^1, s' \in S^1$ such that $t'(txs)s' = x$. If $t'tx \,\mathcal{L}\, x$ then $tx \,\mathcal{L}\, x$, thus $(t'tx, x) \notin \mathcal{L}$ and dually $(x, xss') \notin \mathcal{R}$. Let (A, U, B) be the bicyclic biact over $\{a\}, \{u\}, \{b\}$ and (T', X', S') be the subact of (T, X, S) with $T' =< t't >$ the subsemigroup of T generated by $t't$, $S' =< ss' >$ the subsemigroup of S generated by ss' and $X' = T'xS'$. The map $(\phi, f, \psi) : (A, U, B) \to (T', X', S')$ defined for any $p, q > 0$ by $\phi(a^p) = (t't)^p$, $f(u) = x$, $f(a^pu) = (t't)^px$, $f(ub^q) = x(ss')^q$ and $\psi(b^q) = (ss')^q$ is a surjective morphism of biacts. Assume that ϕ is not injective. Then there exist $0 < p < p'$ such that $(t't)^p = (t't)^{p'}$ and consequently there exists p'' such that $(t't)^{p''}$ is idempotent. It follows that

$$(t't)^{p''-1}(t't)x = (t't)^{p''}x = ((t't)^{p''})^2x(ss')^{p''} = (t't)^{p''}x(ss')^{p''} = x$$

and $t'tx \,\mathcal{L}\, x$, which is absurd. Thus ϕ is injective. Dually ψ is injective. The same arguments (on translations of X') give that f is also injective, and finally (ϕ, f, ψ) is a bijective morphism, whence an isomorphism. □

Example 3.20. Let X be a finite (T, S)-biact. Then it has (at least one) minimal subact which is $\mathcal{J}$-simple and completely stable.

3.3 Schützenberger groups

Let $\mathbf{X} = (T, X, S)$ be a given semigroup biact (equivalently let ${}_TX_S$ be a given (T, S)-biact).

Let $H = R \cap L$ and $K = R \cap L'$ be two $\mathcal{H}-$classes in the same $\mathcal{R}$-class. The left stabilizer of H defined by $stab_l(H) = \{t \in T^1 | tH = H\}$ is a submonoid of T^1, equal to $\{t \in T^1 | tH \subseteq H\}$ by Green's lemma 3.6. It also coincides with $stab_l(K) = \{t \in T^1 | tK = K\}$ by Corollary 3.9. The elements of $stab_l(H) = stab_l(K)$ define a set of left translations of R (resp. of H) $\Delta_l(H) = \{\delta_t \in \mathcal{T}(R), t \in stab_l(H)\}$ (resp. $\delta_{l,H}(H) = \{\delta_t \in \mathcal{T}(H), t \in stab_l(H)\}$), which is a group by Green's lemma. We call $\Delta_l(H)$ the left Schützenberger group of H, by analogy with the classical case [25] (but in contrast, we consider translations of R rather than translations of H, that is $\Delta_l(H)$ instead of its isomorphic group $\Delta_{l,H}(H)$). This group acts freely and transitively on H on the left, and consequently H is isomorphic (as a left $\Delta_l(H)$-act) to $\Delta_l(H)$. In particular it acts simply transitively on H, that is for any $x, y \in H$ there exists a unique $f \in \Delta_l(H)$ such that $f(x) = y$.

Lemma 3.21. *Let $x \in X$. Pose $H = H_x$. Then the function $\theta : H \to \Delta_l(H)$ that maps $y = tx \mapsto \delta_t$ is a bijection, with reciprocal $\pi : \delta_t \mapsto tx$. In particular $\Delta_l(H)$ acts freely and transitively on H (by $\delta_t.h = th$).*

Proof. Let $x \in X$ and $t, t' \in T$ such that $tx = t'x \in H_x = H$. By cancellation, if $z \,\mathcal{H}\, x$ then $tz = t'z$ and $\delta_t = \delta_{t'}$. Also $\delta_t \in \Delta_l(H)$ by Lemma 3.6. Finally, by definition of $\Delta_l(H)$ and Lemma 3.6, and as $\delta_t : x \mapsto tx$ then π is well-defined from $\Delta_l(H)$ to H. It is the reciprocal of θ by construction. Finally let $y, z \in H$. Then $(\theta(z)\theta(y)^{-1}).y = (\theta(z)\theta(y)^{-1}).(\theta(y)x) = \theta(z).x = z$ and the action is transitive. It is free since $\delta_t.y = y$ means $ty = y$ and by cancellation, $tz = z$ so that $\delta_t = 1$, the identity of the group. □

By construction, the left Schützenberger groups of H and K in the same $\mathcal{R}$-class R are equal, that is, $\Delta_l(R \cap L) = \Delta_l(R \cap L')$. By duality, we define the right Schützenberger group of $H = R \cap L$ as the group of right translations of L $\Delta_r(H) = \{\tau_s \in \mathcal{T}(L)^{op}, s \in stab_r(H)\}$.

We now prove that right Schützenberger groups and left Schützenberger groups are isomorphic (but the isomorphism is not canonical).

Lemma 3.22. *Let $x \in X$. Pose $H = H_x$. Then the function $\varphi_H : \Delta_l(H) \to \Delta_r(H)$ that maps $\delta_t \mapsto \tau_s$ where $tx = xs$ is a group isomorphism, with reciprocal $\phi_H : \Delta_r(H) \to \Delta_l(H)$ that maps $\tau_s \mapsto \delta_t$.*

Proof. Let $x \in X$ and $t \in stab_l(H)$. Then $tx \,\mathcal{H}\, x$ and exists $s \in S^1$, $tx = xs$. As $xs \,\mathcal{H}\, x$ then $s \in stab_r(H)$ by Green's lemma. Let $s' \in S^1$ such that $tx = xs'$, and let $y \,\mathcal{L}\, x$. As $xs = xs'$ then by cancellation $ys = ys'$, and $\tau_s = \tau'_s$ as functions on $L = L_x$, and φ is well-defined. Let now $t, t' \in stabl_l(H)$, and let s, s' such that $tx = xs$ and $t'x = xs'$. Then $t'tx = t'xs = xs's$, and φ is a morphism. By duality $\phi_H : \Delta_r(H) \to \Delta_l(H)$ that maps $\tau_s \mapsto \delta_t$ where $tx = xs$ is also well-defined and a morphism, and the two morphisms are reciprocal. □

3.4 Coherent cross sections

Consider a $\mathcal{D}$-class D of the (T, S)-biact ${}_TX_S$. Denote by I the set of $\mathcal{R}$-classes and Λ the set of $\mathcal{L}$-classes of D, and assume that I and Λ contain an element denoted by 1. For $i \in I$ and $\lambda \in \Lambda$ we pose $H_{i\lambda} = i \cap \lambda$ and denote it by $i\lambda$ for short. A family $\{x_{i\lambda} \in i\lambda | i \in I, \lambda \in \Lambda\}$ is called a *cross-section.* Any element $x \in D$ may be uniquely written $x = f(x_{i\lambda}) = [x_{i\lambda}]g$ with $f \in \Delta_l(i\lambda)$ and $g \in \Delta_r(i\lambda)$.

The cross-section is called coherent if the diagram of Figure 1 defined by the isomorphisms of Lemma 3.22 is commutative, or equivalently if we have the equality $\varphi_{i\lambda} = \varphi_{j\lambda}\phi_{j\mu}\varphi_{i\mu}$ for all $i, j \in I$ and $\lambda, \mu \in \Lambda$.

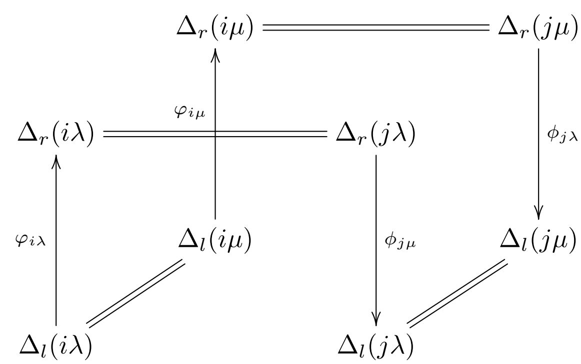

Figure 1: Coherence of a cross-section

The following result was obtained by Grillet [10] in the case of semigroups.

Theorem 3.23. *Any family* $\{x_{1\lambda} \in 1\lambda, x_{i1} \in i1 | i \in I, \lambda \in \Lambda\}$ *can be completed in a coherent cross-section.*

Proof. Let $\{x_{1\lambda} \in 1\lambda, x_{i1} \in i1 | i \in I, \lambda \in \Lambda\}$. By definition of $\mathcal{L}$ and $\mathcal{R}$, for any $i \in I$ and $\lambda \in \Lambda$ there exist $t_i, t_i^{'} \in T^1$ and $s_\lambda, s_\lambda^{'} \in S^1$ such that $x_{i1} = t_i x_{11},\ x_{11} = t_i^{'} x_{i1}$ and $x_{1\lambda} = x_{11} s_\lambda,\ x_{11} = x_{1\lambda} s_\lambda^{'}$. Set $x_{i\lambda} = t_i x_{11} s_\lambda = t_i x_{1\lambda} = x_{i1} s_\lambda$. Then computations similar to the semigroup case [10] give that $\{x_{i\lambda} | i \in I, \lambda \in \Lambda\}$ is a coherent cross-section. First, it holds that $x_{j\lambda} s_\lambda^{'} s_\mu = x_{j\mu}$, and $t_i t_j^{'} x_{j\mu} = x_{i\mu}$ for any $i, j \in I$ and $\lambda, \mu \in \Lambda$. Fix $i, j \in I$ and $\lambda, \mu \in \lambda$ and let $\delta_t \in \delta(i\lambda)$. Let $u, w \in S^1, v \in T^1$ such that $t x_{i\mu} = x_{i\mu} u, v x_{j\mu} = x_{j\mu} u, v x_{j\lambda} = x_{j\lambda} w$ so that $\varphi_{j\lambda} \phi_{j\mu} \varphi_{i\mu}(\delta_t) = \tau_w$. It holds that

$$\begin{aligned}
t x_{i\lambda} &= t t_i x_{11} s_\lambda = t t_i x_{11} s_\mu s_\mu^{'} s_\lambda \\
&= t x_{i\mu} s_\mu^{'} s_\lambda = x_{i\mu} u s_\mu^{'} s_\lambda \\
&= t_i t_j^{'} x_{j\mu} u s_\mu^{'} s_\lambda = t_i t_j^{'} v x_{j\mu} s_\mu^{'} s_\lambda \\
&= t_i t_j^{'} v x_{j\lambda} s_\lambda^{'} s_\mu s_\mu^{'} s_\lambda = t_i t_j^{'} x_{j\lambda} w \\
&= x_{i\lambda} w
\end{aligned}$$

so that $\varphi_{i\lambda}(\delta_t) = \tau_w$. □

Figure 2 describes the construction of the coherent cross-section.

We now interpret cross-sections in terms of biaction of a group. Let $H = 11$. As for any $i, j \in I$ and $\lambda, \mu \in \Lambda$ we have the equality $\Delta_l(i\lambda) = \Delta_l(i\mu)$ and $\Delta_r(i\lambda) = \Delta_r(j\lambda)$, we can define group isomorphisms $a_{i\lambda} = \phi_{i\lambda} \circ \varphi_{1\lambda} : \Delta_l(11) \to \Delta_l(i\lambda)$ and $b_{i\lambda} = \varphi_{i\lambda} \circ \phi_{i1} : \Delta_r(11) \to \Delta_r(i\lambda)$. The group $G = \Delta_l(11)$ acts on D on the left by $g \cdot x = a_{i\lambda}(g)(x)$ for any $x \in i\lambda$. This is a group action since $g' \cdot (g \cdot x) = g' \cdot a_{i\lambda}(g)(x)$ with $a_{i\lambda}(g)(x) \in i\lambda$, hence $g' \cdot (g \cdot x) = a_{i\lambda}(g')(a_{i\lambda}(g)(x)) = a_{i\lambda}(g'g)(x)$ since $a_{i\lambda}$ is an isomorphism. Dually

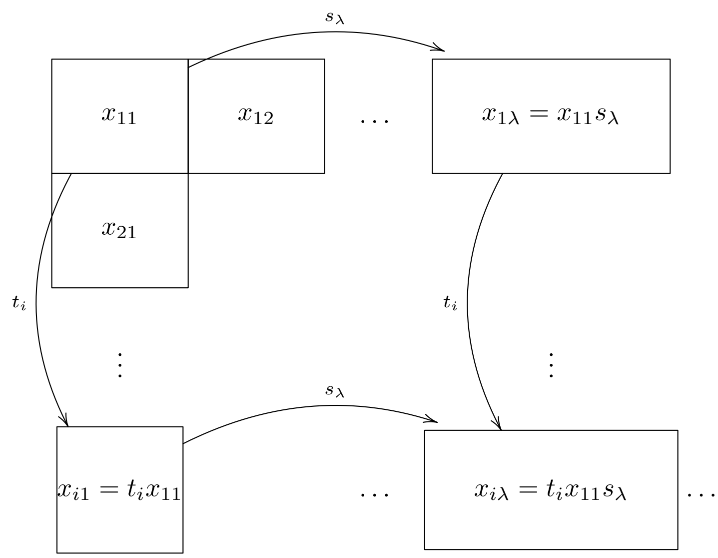

Figure 2: Construction of a coherent cross-section

$\Delta_r(11)$ acts on D on the right, and since $\Delta_l(11)$ and $\Delta_r(11)$ are isomorphic, $G = \Delta_l(11)$ acts also on D on the right by $x \odot g = [x]b_{i\lambda}(\varphi_{11}(g))$.

These group actions are free on D, and always compatible since they are left and right translations of a biact.

Lemma 3.24. *The left and right actions of G (defined as above by a given cross-section) are free on D, and always compatible.*

Proof. By construction, the Schützenberger groups act freely on their respective $\mathcal{H}$-class, so that the group G acts freely on D. Let $g, g' \in G$ and $x \in i\lambda$. Then exists $t \in stab_l(i\lambda), s \in stab_r(i\lambda)$, $a_{i\lambda}(g') = \delta_t$ and $b_{i\lambda}(g) = \tau_s$. Thus $g' \cdot (x \odot g) = t(xs) = (tx)s = (g' \cdot x) \odot g$. ☐

Lemma 3.25. *The cross section is coherent if and only if*

$$(\forall x \in i\lambda, g \in G)\ x \odot g = [x]\varphi_{1\lambda}(g).$$

Proof. Let $i, j \in I$ and $\lambda, \mu \in \Lambda$. Let also $x \in i\lambda$ and $g \in G = \Delta_l(11)$.
If the cross-section is coherent, then $\varphi_{i\lambda} = \varphi_{j\lambda}\phi_{j\mu}\varphi_{i\mu}$ and $x \odot g = b_{i\lambda}(\varphi_{11}(g)) = \varphi_{i\lambda}\phi_{i1}\varphi_{11}(g) = [x]\varphi_{1\lambda}(g)$.
Conversely, if the equality holds then $\varphi_{i\lambda}\phi_{i1}\varphi_{11} = \varphi_{1\lambda}$ and since the maps are bijective, then $\varphi_{i\lambda} = \varphi_{1\lambda}\phi_{11}\varphi_{i1}$. Also $\varphi_{i\mu} = \varphi_{1\mu}\phi_{11}\varphi_{i1}$, $\varphi_{j\lambda} = \varphi_{1\lambda}\phi_{11}\varphi_{j1}$ and $\phi_{j\mu} = \phi_{j1}\varphi_{11}\phi_{1\mu}$. We finally get

$$\begin{aligned} \varphi_{j\lambda}\phi_{j\mu}\varphi_{i\mu} &= \varphi_{1\lambda}\phi_{11}\varphi_{j1}\phi_{j1}\varphi_{11}\phi_{1\mu}\varphi_{1\mu}\phi_{11}\varphi_{i1} \\ &= \varphi_{1\lambda}\phi_{11}\varphi_{i1} \\ &= \varphi_{i\lambda} \end{aligned}$$

□

Coherence expresses that the actions of G are actually compatible (in the sense given below) with the actions of T and S.

Proposition 3.26. *If the cross-section is coherent then the monoid biact* (G, D, G) *satisfies for any* $g \in G$*:*

1. $g \cdot (xs) = (g \cdot x)s$ *for any* $s \in S, x \in D$ *such that* $x \,\mathcal{R}\, xs$*;*
2. $(tx) \odot g = t(x \odot g)$ *for any* $t \in T, x \in D$ *such that* $tx \,\mathcal{L}\, x$*;*
3. $g \cdot x_{i\lambda} = x_{i\lambda} \odot g$ *for all* $i \in I, \lambda \in \Lambda$.

Proof. Let $g \in G$, $t \in T, s \in S, x \in i\lambda$ such that $tx \,\mathcal{L}\, x \,\mathcal{R}\, xs$. Pose $j = R_{tx}$ and $\mu = L_{xs}$.

1. By definition $(g \cdot x)s = \phi_{i\lambda}\varphi_{1\lambda}(g)(x)s$, and $g \cdot (xs) = \phi_{i\mu}\varphi_{1\mu}(g)(xs)$. As the cross-section is coherent, then the left translations $\phi_{i\lambda}\varphi_{1\lambda}(g)$ and $\phi_{i\mu}\varphi_{1\mu}(g)$ are equal and $(g \cdot x)s = g \cdot (xs)$.
2. By Lemma 3.25 $t(x \odot g) = t[x]\varphi_{1\lambda}(g)$, and $(tx) \odot (g) = [tx]\varphi_{1\lambda}(g)$ with $\varphi_{1\lambda}(g)$ a right translation, so that the two elements are equal.
3. By definition $g \cdot x_{i\lambda} = \phi_{i\lambda}\varphi_{1\lambda}(g)(x_{i\lambda})$. Set $\delta_v = \phi_{i\lambda}\varphi_{1\lambda}(g)$ with $v \in stab_l(i\lambda)$. Then by definition of $\varphi_{i\lambda}$, $vx_{i\lambda} = [x_{i\lambda}]\varphi_{i\lambda}(\delta_v)$ thus $g \cdot x_{i\lambda} = [x_{i\lambda}]\varphi_{i\lambda}\phi_{i\lambda}\varphi_{1\lambda}(g) = [x_{i\lambda}]\varphi_{1\lambda}(g)$. Now since the cross-section is coherent $x_{i\lambda} \odot g = [x_{i\lambda}]\varphi_{1\lambda}(g)$ by Lemma 3.25. This ends the proof.

□

4 Structure of stable, $\mathcal{J}$-simple biacts

4.1 Wreath products, tensor products and related constructions

We start with general constructions on semigroup acts. Semidirect products of semigroups, wreath product of semigroups and left (right) acts, and wreath products of left (right) acts are relatively well-known constructions. We recall them for convenience, and generalize them to the case where *one* of the acts in question is a biact.

Definition 4.1 (Wreath Product)**.** Let M (resp. (M, X), (M, X, N)) be a semigroup (resp. a left semigroup act, a semigroup biact) and (T, Y) (resp. (Z, S)) be a left (resp. right) semigroup act. Then T (resp. S) acts on M^Y (resp. N^Z) by composition on the right (resp. left).

1. The semidirect product $(T, Y) \wr M = T \ltimes M^Y$ with product

$$(t, f)(t', f') = (tt', (f \circ \delta_{t'})f')$$

(where $\delta_t : x \mapsto tx$) is called the wreath product of (T, Y) and M.

2. Dually, the semidirect product $N \wr (Z, S) = N^Z \rtimes S$ with product

$$(g', s')(g, s) = (g'(\tau_{s'} \circ^{op} g), s's)$$

is called the wreath product of N and (Z, S).

3. The semigroup $(T, Y) \wr M$ acts on the left on $Y \times X$ by

$$(t, f) \cdot (y, x) = (ty, f(y)x).$$

The left act $(T, Y) \wr (M, X) = ((T, Y) \wr M, Y \times X)$ is called the wreath product of (T, Y) and (M, X).

4. Dually, the semigroup $N \wr (Z, S)$ acts on the right on $X \times Z$ by

$$(x, z) \odot (g, s) = (x[z]g, zs).$$

The right act $(X, N) \wr (Z, S) = (N \wr (Z, S), X \times Z)$ is called the wreath product of (X, N) and (Z, S).

5. The biact $(T, Y) \wr (M, X, N)$ (resp. $(M, X, N) \wr (Z, S)$) is the 5-uple $((T, Y) \wr M, Y \times X, N, \alpha, \beta)$ (resp.$(M, X \times Z, N \wr (Z, S), \alpha, \beta)$) with actions

$$\alpha\left((t, f), (y, x)\right) = (t, f) \cdot (y, x) = (ty, f(y)x),\ \beta\left((y, x), n\right) = (y, x) \odot n = (y, xn)$$

(resp. $\alpha\left(m, (x, z)\right) = m \cdot (x, z) = (mx, z)$, $\beta\left((x, z), (g, s)\right) = (x, z) \odot (g, s) = (x[z]g, zs)$).

6. Finally, we define $(T, Y) \wr (M, X, N) \wr (Z, S)$ as $((T, Y) \wr M, Y \times X \times Z, N \wr (Z, S), \alpha, \beta)$ with actions

$$\alpha\left((t, f), (y, x, z)\right) = (t, f) \cdot (y, x, z) = (ty, f(y)x, z),$$

$$\beta\left((y, x, z), (g, s)\right) = (y, x, z) \odot (g, s) = (y, x[z]g, zs).$$

The actions are indeed compatible as (for instance in 5.) $((t, f) \cdot (y, x)) \odot n = (ty, f(y)xn) = (t, f) \cdot ((y, x) \odot n)$. We cannot unambiguously define a wreath product $(T, Y, N) \wr (M, Z, S)$, because the left action (of $(T, Y) \wr M$ on $Y \times Z$) and the right action (of $N \wr (Z, S)$ on $Y \times Z$) will not be compatible in general.

The wreath product of left (right) acts is associative up to an isomorphism (see, for example, [11]). Incorporating biacts into the picture, we have, more generally:

Lemma 4.2. *Let (M, X, N) be a semigroup biact, and (T_i, Y_i) (resp. (Z_i, S_i)) be left (resp. right) semigroup acts for $i = 1, 2, 3$. Then*

1. *(Associativity)* $((T_1, Y_1) \wr (T_2, Y_2)) \wr (T_3, Y_3) \sim (T_1, Y_1) \wr ((T_2, Y_2) \wr (T_3, Y_3))$.
2. *(Associativity)* $(Z_1, S_1) \wr ((Z_2, S_2) \wr (Z_3, S_3)) \sim ((Z_1, S_1) \wr (Z_2, S_2)) \wr (Z_3, S_3)$.
3. *(Left action)* $((T_1, Y_1) \wr (T_2, Y_2)) \wr (M, X, N) \sim (T_1, Y_1) \wr ((T_2, Y_2) \wr (M, X, N))$.

4. *(Right action)* $(Z_1, S_1) \wr ((Z_2, S_2) \wr (M, X, N)) \sim ((Z_1, S_1) \wr (Z_2, S_2)) \wr (M, X, N)$.

5. *(Compatibility)* $((T_1, Y_1) \wr (M, X, N)) \wr (Z_1, S_1) \sim (T_1, Y_1) \wr ((M, X, N) \wr (Z_1, S_1))$.

Also $((T_1, Y_1) \wr (M, X, N)) \wr (Z_1, S_1) \sim (T_1, Y_1) \wr (M, X, N) \wr (Z_1, S_1)$ *(defined in Definition 4.1).*

Proof. The proof is just technical, and consists in just computing all the different products. □

Our next lemma studies Green's relations on wreath products of semigroup acts.

Lemma 4.3. *Let* (M, X) *(resp.* (M, X, N)*) be a left semigroup act (resp. a semigroup biact), and* (T, Y) *be a left semigroup act. Let* $x, x' \in X$ *and* $y, y' \in Y$*, and assume that* $M^1 x = Mx$, $M^1 x' = Mx'$, $T^1 y = Ty$, $Y^1 y' = Ty'$*. Then*

1. $(y, x) \,\mathcal{L}\, (y', x')$ *in* $(T, Y) \wr (M, X)$ *(resp.* $(T, Y) \wr (M, X, N)$*) if and only if* $y \,\mathcal{L}\, y'$ *(in* (T, Y)*) and* $x \,\mathcal{L}\, x'$ *(in* (M, X)*, resp. in* (M, X, N)*).*

2. $(y, x) \,\mathcal{R}\, (y', x')$ *in* $(T, Y) \wr (M, X, N)$ *if and only if* $y = y'$ *and* $x \,\mathcal{R}\, x'$ *(in* (M, X, N)*).*

3. $(y, x) \,\mathcal{J}\, (y', x')$ *in* $(T, Y) \wr (M, X, N)$ *if and only if* $y \,\mathcal{L}\, y'$ *(in* (T, Y)*) and* $x \,\mathcal{J}\, x'$ *(in* (M, X, N)*).*

Proof. We prove only the first item. Assume that there exist $t \in T$, $m \in M$ such that $y' = ty$ and $x' = mx$. Let $f \in M^Y$ be the constant function m. Then $(t, f) \cdot (y, x) = (ty, f(y)x) = (ty, mx = (y', x')$. Thus (by symmetry) $y \,\mathcal{L}\, y'$ and $x \,\mathcal{L}\, x' \Rightarrow (y, x) \,\mathcal{L}\, (y', x')$. Conversely, assume that there exist $t \in T$, $f \in M^I$ such that $(t, f) \cdot (y, x) = (y', x')$. Then $ty = y'$ and $f(y)x = x'$, and by symmetry $(y, x) \,\mathcal{L}\, (y', x') \Rightarrow y \,\mathcal{L}\, y'$ and $x \,\mathcal{L}\, x'$. □

The conclusion may fail without the equality of left ideals, as shows next example.

Example 4.4. Let $T =< t >$ be the free semigroup of 1 generator t, and G be a (non trivial) group. Form $(T, \underline{T}) \wr (G, \underline{G})$ the wreath product of the left semigroup acts $(T, \underline{T})$ and $(G, \underline{G})$. Let $g \neq h \in \underline{G}$. Then $g \,\mathcal{L}\, h$ in $(G, \underline{G})$ and $t \,\mathcal{L}\, t$ in $(T, \underline{T})$, but $(t, g) \neq (t, h)$ are not $\mathcal{L}$-related in $(T, \underline{T}) \wr (G, \underline{G})$. Indeed, any element of $T \ltimes G^{\underline{T}}$ is of the form (t^p, f) with $p > 0$ and $(t^p, f) \cdot (t, g) = (t^{p+1}, f(t)g) \neq (t, h)$ since $p + 1 > 1$.

We deduce the result for right semigroup acts by duality. As a corollary, and using Lemma 3.2, we get results concerning $\mathcal{J}, \mathcal{L}, \mathcal{R}$-simplicity and stability of wreath products of semigroup acts and biacts.

Corollary 4.5. *Let* (M, X) *(resp.* (M, X, N)*) be a left semigroup act (resp. a semigroup biact), and* (T, Y) *(resp.* (Z, S)*) be a left (resp. right) semigroup act. Then*

1. $(T,Y)\wr(M,X)$ *(or* $(T,Y)\wr(M,X,N)$*) is* $\mathcal{L}$*-simple (resp. left stable) if and only if* (T,Y) *and* (M,X) *(or* (M,X,N)*) are* $\mathcal{L}$*-simple (resp. left stable).*

2. $(X,N)\wr(Z,S)$ *(or* $(M,X,N)\wr(Z,S)$*) is* $\mathcal{R}$*-simple (resp. right stable) if and only if* (X,N) *(or* (M,X,N)*) and* (Z,S) *are* $\mathcal{R}$*-simple (resp. right stable).*

3. $(T,Y)\wr(M,X,N)$ *is* $\mathcal{J}$*-simple (resp. stable) if and only if* (T,Y) *is* $\mathcal{L}$*-simple (resp. left stable) and* (M,X,N) *is* $\mathcal{J}$*-simple (resp. stable).*

4. $(M,X,N)\wr(Z,S)$ *is* $\mathcal{J}$*-simple (resp. stable) if and only if* (Z,S) *is* $\mathcal{R}$*-simple (resp. right stable) and* (M,X,N) *is* $\mathcal{J}$*-simple (resp. stable).*

5. $((T,Y)\wr(M,X,N))\wr(Z,S)$ *is* $\mathcal{J}$*-simple (resp. stable) if and only if* (T,Y) *is* $\mathcal{L}$*-simple (resp. left stable),* (Z,S) *is* $\mathcal{R}$*-simple (resp. right stable) and* (M,X,N) *is* $\mathcal{J}$*-simple (resp. stable).*

We now apply the previous results on wreath products to a specific act, the full transformation semigroup on a set, and a specific biact, a monoid acting on itself by left and right multiplication (or even more specifically in some cases, a group). Precisely, let I be a set and M be a monoid. Let $\mathcal{T}(I)=(I^I,\circ)$ be the full transformation semigroup on I with composition of functions. The monoid $\mathcal{T}(I)$ acts on I on the left so that $(\mathcal{T}(I),I)$ is a left monoid act. Dually, let Λ be a set. Then $(\Lambda,\mathcal{T}^{op}(\Lambda))$ is a right monoid act. Let M be a monoid (we will mainly consider the case $M=G$ is a group afterwards). Then M acts on itself on the left and on right by multiplication in a compatible way, so that $(M,\underline{M},M)$ is a monoid biact (where, according to our conventions, $\underline{M}$ is the underlying set of M). We can then form the monoid biact

$$\begin{aligned}\boldsymbol{FB(I,M,\Lambda)} &= (\mathcal{T}(I),I)\wr(M,\underline{M},M)\wr(\Lambda,\mathcal{T}^{op}(\Lambda))\\ &= \left(\mathcal{T}(I)\ltimes M^I, I\times\underline{M}\times\Lambda, M^\Lambda\rtimes\mathcal{T}^{op}(\lambda)\right).\end{aligned}$$

We will call this monoid biact the *Full Biact over (*I*,* M*,* Λ*)*.

The following lemma is straightforward:

Lemma 4.6. *Let* I,Λ *be sets. Then* $(\mathcal{T}(I),I)$ *is* $\mathcal{L}$*-simple and* $(\Lambda,\mathcal{T}^{op}(\Lambda))$ *is* $\mathcal{R}$*-simple. In particular, they are respectively left and right stable.*

Proposition 4.7. *Let* I,Λ *be sets and* G *be a group. Then the monoid biact* $\boldsymbol{FB(I,G,\Lambda)}=(\mathcal{T}(I),I)\wr(G,\underline{G},G)\wr(\Lambda,\mathcal{T}^{op}(\Lambda))$ *is* $\mathcal{J}$*-simple and stable.*

Proof. As $(\mathcal{T}(I),I)$ is $\mathcal{L}$-simple by Lemma 4.6 and left stable and $(G,\underline{G},G)$ is $\mathcal{J}$-simple and stable, then $(\mathcal{T}(I),I)\wr(G,\underline{G},G)$ is $\mathcal{J}$-simple and stable by Corollary 4.5. Still by Corollary 4.5, as also $(\Lambda,\mathcal{T}(\Lambda))$ is $\mathcal{R}$-simple and right stable then $\boldsymbol{FB(I,G,\Lambda)}$ is $\mathcal{J}$-simple and stable. □

The set of $\mathcal{R}$-classes of $\boldsymbol{FB(I,G,\Lambda)}$ is in bijection with I, its set of $\mathcal{L}$-classes is in bijection with Λ and each $\mathcal{H}$-class is equipotent with $\underline{G}$. An interpretation of this biact is as follows. Let $P=(P_{i\lambda})$ be any sandwich matrix over G and form the (completely simple) Rees matrix semigroup $C = \mathcal{M}(I, G, \Lambda, P)$. From [22] and [7], see also Proposition 4.1 in [9], the semigroup $L(C)$ of left translations of C ($l(xy) = l(x)y\,(\forall x, y \in C)$) is isomorphic to $(\mathcal{T}(I), I) \wr G = \mathcal{T}(I) \ltimes G^I$, and dually the semigroup $R(C)$ of right translations of C is isomorphic to $G \wr (\mathcal{T}^{op}(\Lambda), \Lambda) = G^\lambda \rtimes \mathcal{T}(I)$ so that $\boldsymbol{FB(I, G, \Lambda)} \sim (L(C), \underline{C}, R(C))$, biact of left and right translations acting on the completely simple semigroup $C = \mathcal{M}(I, G, \Lambda, P)$. Conversely, by Rees Theorem any biact of the form $(L(C), \underline{C}, R(C))$ for some completely simple semigroup C is isomorphic to the biact $\boldsymbol{FB(I, G, \Lambda)}$ for some sets I, Λ and group G.

It is proved in [11] Theorem 6.18, see also [26], that wreath products of the type $(\mathcal{T}(X), X) \wr (M, \underline{M})$ where M is a monoid can be described by means of endomorphisms of free M-acts. Precisely, denote by $XM = (X \times \underline{M}, M, \beta)$ the free right M-act with basis X and action $\beta\,((x, m), m') = (x, mm')$. By $End(XM)$ we mean $Hom((X \times \underline{M})_M, (X \times \underline{M})_M)$ in the category **Right *M*-act**. The monoid $End(XM)$ acts on $X \times \underline{M}$ in a canonical way.

Theorem 4.8 ([11] Theorem 6.18)**.**
The left acts $(\mathcal{T}(X), X) \wr (M, \underline{M})$ and $(End(XM), X \times \underline{M})$ are isomorphic.

Indeed, M-endomorphisms of XM are in bijection with functions from X to M. In the same spirit, it is not difficult to make the following identification:

Proposition 4.9. *Let I, Λ be sets and M be a monoid. Then $End(IM)$ (resp. $End^{op}(M\Lambda)$) acts on $(I \times \underline{M} \times \Lambda)$ on the left (resp. on the right) by $\phi \cdot (i, m, \lambda) = \phi(i, m)\lambda$ (resp . $(i, m, \lambda) \odot \psi = i[m, \lambda]\psi$) and these actions are compatible. It holds that*

$$\boldsymbol{FB(I, M, \Lambda)} \sim (End(IM), I \times \underline{M} \times \Lambda, End^{op}(M\Lambda))$$

We will write $\boldsymbol{EB(I, M, \Lambda)} = (End(IM), I \times \underline{M} \times \Lambda, End^{op}(M\Lambda))$, *Endomorphism Biact over* (I, M, Λ).

We can use the following classical notation to describe this biact. Write any element $x = (i, m, \lambda)$ by juxtaposition, $x = im\lambda$. Then $\phi \cdot (im\lambda) \odot \psi = \phi(i1)m[1\lambda]\psi$ for $\phi \in End(IM)$, $\psi \in End^{op}(M\Lambda)$.

We can also describe this biact in terms of tensor products when $M = G$ is a group (the situation we will encounter later). Consider the two biacts $(End(IG), I \times \underline{G}, G)$ and $(G, \underline{G} \times \Lambda, End^{op}(G\Lambda))$. Then some calculations prove that their tensor product is isomorphic to $(End(IG), I \times \underline{G} \times \Lambda, End^{op}(G\Lambda))$:

$$(End(IG), I \times \underline{G}, G) \otimes (G, \underline{G} \times \Lambda, End^{op}(G\Lambda) \sim (End(IG), I \times \underline{G} \times \Lambda, End^{op}(G\Lambda))\,.$$

Indeed, by introducing *tossings* [11] in the picture, we can prove that $(i, g) \otimes (h, \lambda) = (i', g') \otimes (h', \lambda')$ if and only if $i = i', gh = g'h'$ and $\lambda = \lambda'$ so that $(i, g) \otimes (h, \lambda) \mapsto (i, gh, \lambda)$ is a bijection from $(I \times \underline{G}, G) \otimes (G, \underline{G} \times \Lambda)$ onto $I \times \underline{G} \times \lambda$ that respects the left and right actions.

We can finally represent this biact as square matrices acting on (non-square) matrices over a monoid with zero (denoted by $\star$ afterwards). Indeed, define $\mathcal{M}^c_{I,I}(M)$ as the $I \times I$ matrices with coefficients in the monoid $M \bigcup \{\star\}$ (with $\star$ the zero of the monoid) such that each column contains exactly one coefficient in M, and the others $\star$. Such matrices are sometimes called column-monomial matrices (over M). Define a partial sum on $M \bigcup \{\star\}$ by $\star + \star = \star$ and $\star + m = m$ for any $m \in M$, and the product on $\mathcal{M}_{I,I}(M)$ by the classical formula

$$A \times B(i,j) = \sum_{k \in I} A(i,k)B(k,j).$$

Then $End(IM) \sim \mathcal{M}_{I,I}(M)$ with isomorphism given by $\phi \mapsto A_\phi$, with $A_\phi(i,j) = m$ if $\phi(j1) = im$ and $\star$ otherwise. Dually, we can define $\mathcal{M}^r_{\Lambda,\Lambda}(M)$ (each row contains exactly one coefficient in M *i.e.* row-monomial matrices) and identify $End^{op}(M\Lambda) \sim \mathcal{M}^r_{\Lambda,\Lambda}(M)$ with isomorphism given by $\psi \mapsto B_\psi$, with $B_\psi(\lambda,\mu) = m$ if $[1\lambda]\psi = m\mu$. Finally define $\mathcal{M}^s_{I,\Lambda}(M)$ as the $I \times \Lambda$ matrices with coefficients in $M \bigcup \{\star\}$ such that exactly one coefficient is in M, in bijection with $(I \times M \times \Lambda)$ by $x = (i,m,\lambda) \mapsto C_x$ such that $C_x(i,\lambda) = m$ and ($\star$ otherwise). Matrix multiplication (on the left and on the right) define a monoid biact

$$\boldsymbol{\mathcal{M}(I,M,\Lambda)} = (\mathcal{M}^c_{I,I}(M), \mathcal{M}_{I,\Lambda}(M), \mathcal{M}^r_{\Lambda,\Lambda}(M)).$$

Proposition 4.10. *With the above notations, it holds that*

$$\boldsymbol{EB(I,M,\Lambda)} \sim \boldsymbol{\mathcal{M}(I,M,\Lambda)}$$

Proof. We only have to check that the previous isomorphisms preserve the biaction, or equivalently the left and right actions. We consider the left action (the right action is dual). Let $\phi \in End(IM)$ and $x = (i,m,\lambda)$ in $I \times M \times \Lambda$. Then $\phi \cdot x = \phi(i1)m\lambda = jm'm\lambda$ for some $j \in I, m' \in M$. Define A_ϕ and C_x as above. Then $[A_\phi C_x](k,l) = \star$ if $l \neq \lambda$, and $[A_\phi C_x](k,\lambda) = A_\phi(k,i)m = \star$ if $k \neq j$ and $m'm$ if $k = j$. This ends the proof. □

Example 4.11. We illustrate (some of) the various representations on a toy example. Let $I = \{1,2\}$ and $M = < m >^1$ be the monoid generated by m. An element of $\mathcal{T}(I) \ltimes M^I$ is a function from I to $I \times M$. Consider for instance f with $f(1) = (2,m)$ and $f(2) = (2,m^3)$. We associate to f the M-endomorphims of IM ϕ_f defined by $\phi_f(1m^k) = 2m^{k+1}$ and $\phi_f(2m^k) = 2m^{k+3}$, and then to $\phi = \phi_f$ the matrix

$$A_\phi = \begin{pmatrix} \star & \star \\ m & m^3 \end{pmatrix}.$$

Let finally $x = (1,m^2)$. Then $f \cdot x = (f(1), f(1)m^2) = (2,m^3)$. Equivalently $\phi \cdot (1m^2) = 2m^3$ and

$$A_\phi \begin{pmatrix} m^2 \\ \star \end{pmatrix} = \begin{pmatrix} \star & \star \\ m & m^3 \end{pmatrix} \begin{pmatrix} m^2 \\ \star \end{pmatrix} = \begin{pmatrix} \star \\ m^3 \end{pmatrix}.$$

4.2 Stable, $\mathcal{J}$-simple biacts

We are now in position to produce a structure theorem for stable, $\mathcal{J}$-simple semigroup biacts.

Let $\mathbf{X} = (T, X, S)$ be a stable, $\mathcal{J}$-simple semigroup biact. Let $\{x_{i\lambda}, i \in I, \lambda \in \Lambda\}$ be a coherent cross-section of X and set $G = H_{11}$. Any element $x \in i\lambda$ may be uniquely written as (i, g, λ) with g the unique element of G such that $g \cdot x_{i\lambda} = x = x_{i\lambda} \odot g$, so that $X \sim I \times G \times \Lambda$. We now describe the action of $t \in T$ on x.

Lemma 4.12. *Let $i \in I$, $t, t' \in T$ and $x \in i$. Pose $j = \mathcal{R}(tx_{i1})$ the $\mathcal{R}$-class of tx_{i1}. Then $tx \in j$ and $t'tx \,\mathcal{R}\, t'x_{j1}$.*

Proof. By stability, $tx_{i1} \,\mathcal{L}\, x_{i1}$ and by Green's Lemma 3.6 left translation by t maps the $\mathcal{R}$-class i to the $\mathcal{R}$-class j. By the same arguments, left translation by t' maps the $\mathcal{R}$-class j to a $\mathcal{R}$-class and $t'(tx) \,\mathcal{R}\, t'x_{j1}$. □

We locate the product tx in its $\mathcal{H}$-class as follows. Let $x \in i\lambda$, $x \sim (i, g, \lambda)$ (or equivalently $x = x_{i\lambda} \odot g$) and $t \in T$. By stability $tx \,\mathcal{L}\, x$, and by Lemma 4.12 $tx \,\mathcal{R}\, tx_{i1}$ so that $tx \sim (j, g', \lambda)$ with $j = \mathcal{R}(tx_{i1})$ and $g' \in G$ the unique element such that $tx = x_{j\lambda} \odot g'$. By Proposition 3.26 $tx = t(x_{i\lambda} \odot g) = (tx_{i\lambda}) \odot g$. As $tx_{i\lambda} \in j\lambda$ then exists a unique $g_m^\lambda \in G$, $g_t^\lambda \cdot x_{j\lambda} = tx_{i\lambda} = x_{j\lambda} \odot g_t^\lambda$, and finally $g' = g_t^\lambda g$, and $tx \sim (\mathcal{R}(tx_{i1}), g_t^\lambda g, \lambda)$.

We end up with a family of functions ϕ^λ from T to $(I \times G)^I \sim I^I \times G^I$ which send $t \in T$ to the function $i \mapsto (\mathcal{R}(tx_{i1}), g_t^\lambda)$ where g_t^λ is the unique solution to $g_t^\lambda \cdot x_{j\lambda} = tx_{i\lambda} = x_{j\lambda} \odot g_t^\lambda$. We finally show that these functions are equal (independent of λ). By construction of the coherent cross-section (Theorem 3.23) there exists $s_\lambda \in S^1$ such that $x_{k\lambda} = x_{k1}s_\lambda$ for all $k \in I$. As $g_t^1 \cdot x_{j1} = tx_{i1} = x_{j1} \odot g_t^1$ then $(g_t^1 \cdot x_{j1})s_\lambda = (tx_{i1})s_\lambda$ and by Proposition 3.26, $g_t^1 \cdot (x_{j\lambda}) = g_t^1 \cdot (x_{j1}s_\lambda) = tx_{i1}s_\lambda = tx_{i\lambda}$. Thus ϕ^λ is independent of λ. We denote this function by ϕ afterwards.

Lemma 4.13. *Function $\phi : T \to (\mathcal{T}(I), I) \wr G$ is a morphism of semigroups such that the left act $(\phi(T), I \times \underline{G})$ is $\mathcal{L}$-simple. It is one-to-one if and only if T acts faithfully on X.*

Proof. Let $t, t' \in M$ and pose $\phi(t) = (\tau, f), \phi(t') = (\tau', f')$. Let $i \in I$. By definition of the wreath product, $\phi(t)\phi(t')(i) = (\tau, f)(\tau', f') = (\tau \circ \tau', (f \circ \tau')f')$. Pose $j = \tau'(i) = \mathcal{R}(t'x_{i1})$ and $k = \tau(j) = \mathcal{R}(tx_{j1})$. Pose also $g' = f'(i)$ the solution to $g' \cdot x_{j1} = t'x_{i1} = x_{j1} \odot g'$ and $g = f(j)$ the solution to $g \cdot x_{k1} = tx_{j1} = x_{k1} \odot g$. As $\mathcal{R}$ is a right congruence and $t'x_{i1} \,\mathcal{R}\, x_{j1}$ then $tt'x_{i1} \,\mathcal{R}\, tx_{j1}$ and $tt'x_{i1} \in k$. Pose $g'' = gg'$. Then

$$\begin{aligned} x_{k1} \odot gg' &= (x_{k1} \odot g)g' \\ &= (tx_{j1}) \odot g' \\ &= t(x_{j1} \odot g') \text{ by Proposition 3.26} \\ &= t(t'x_{i1}) \end{aligned}$$

and finally, $\phi(tt') = \phi(t)\phi(t')$.
Let $i, j \in I$ and $g, h \in G$. As $x_{i1} \,\mathcal{L}\, x_{j1} \odot hg^{-1}$ then there exists $t \in T^1$ such that $tx_{i1} = x_{j1} \odot hg^{-1}$. If $t = 1$ then $i = j$ and $hg^{-1} = 1$ (the action is free) and $(i, g) = (j, h)$ are indeed $\mathcal{L}$-related. So assume $t \in T$. By definition of ϕ, it holds that $\phi(t)(i) = (j, hg{-}1)$ and $\phi(t) \odot (i, g) = (j, hg^{-1}g) = (j, h)$. This proves that $(\phi(T), I \times \underline{G})$ is $\mathcal{L}$-simple.
Finally let $x \in i\lambda$, $x = x_{i\lambda} \odot g$, and let $t, t' \in T$. Then $tx = t'x \Leftrightarrow (tx_{i\lambda}) \odot g = (t'x_{i\lambda}) \odot g$ by Proposition 3.26, and since G is a group this happens if and only if $tx_{i\lambda} = t'x_{i\lambda}$ if and only if $tx_{i1} = t'x_{i1}$, that is $\phi(t)(i) = \phi(t')(i)$. This ends the proof. □

Theorem 4.14. *Let* $\boldsymbol{X} = (T, X, S)$ *be a faithful, stable,* $\mathcal{J}$*-simple semigroup biact. Then there exist two sets* I, Λ*, a group* G *and a subact* $(T_I, I \times \underline{G} \times \Lambda, S_\Lambda)$ *of* $\boldsymbol{FB}(I, G, \Lambda)$ *such that:*

1. $(T_I, I \times \underline{G})$ *is* $\mathcal{L}$*-simple;*
2. $(\underline{G} \times \Lambda, S_\Lambda)$ *is* $\mathcal{R}$*-simple;*
3. $(T, X, S) \sim (T_I, I \times \underline{G} \times \Lambda, S_\Lambda)$*.*

Conversely, any semigroup biact of this form is faithful, stable, and $\mathcal{J}$*-simple.*

Proof. Let I (resp. Λ) be the set of $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) of X. Let $\{x_{ij}, (i, \lambda) \in I \times \Lambda\}$ be a coherent cross-section and G the Schützenberger group of 11. Define as previously the map $\Phi = (\phi, f, \psi)$ by

1. For all $t \in T$, $\phi(t) : i \mapsto (j, g_t)$ with $j = \mathcal{R}(tx_{i1})$ and g_t is the unique solution to $g_t \cdot x_{j1} = mx_{i1} = x_{j1} \odot g_t$;
2. For all $x \in X$, $f(x) = (i, g, \lambda)$ with $i = R_x$, $\lambda = L_x$ and g the unique element of G such that $g \cdot x_{i\lambda} = x = x_{i\lambda} \odot g$;
3. For all $s \in S$, $\psi(s) : \lambda \mapsto (g_s, \mu)$ with $\mu = \mathcal{L}(x_{1\lambda}s)$ and g_s is the unique solution to $g_s \cdot x_{1\mu} = tx_{1\lambda} = x_{1\mu} \odot g_t$.

By definition of the Schützenberger group f is a bijection, and by Lemma 4.13, $\phi : T \to (\mathcal{T}(I), I) \wr G$ is a injective morphism of semigroups such that the left act $(\phi(T), I \times G)$ is $\mathcal{L}$-simple. Dually, ψ is an injective morphism of semigroups such that the right act $(\underline{G} \times \Lambda, \psi(S))$ is $\mathcal{R}$-simple. Also by Proposition 3.26, and since the biact is stable, $\Phi = (\phi, f, \psi)$ is a morphism of biacts.
The converse is straightforward. □

In this theorem, the set I (resp. Λ) is equipotent with the set of $\mathcal{R}$-classes (resp. of $\mathcal{L}$-classes) of X and G is isomorphic to any Schützenberger group of X. If we drop the faithfulness conditions, then we get that any stable, $\mathcal{J}$-simple semigroup biact admits an epimomorphism onto a biact $(T_I, I \times \underline{G} \times \Lambda, S_\Lambda)$ that satsifes the above conditions, bijective on the second variable.

Regarding embeddings, we have realized $\mathbf{X} = (T, X, S)$ as a subact of $\mathbf{FB}(I, G, \Lambda)$. Equivalently, X can be seen as the underlying set of a completely simple semigroup C, and T (resp. S) as a certain subsemigroup of left (resp. right) translations of C.

Corollary 4.15. *Let $\boldsymbol{X} = (T, X, S)$ be a faithful, stable, $\mathcal{J}$-simple semigroup biact. Then there exist a completely simple semigroup C, and subsemigroups $T_L \trianglelefteq L(C)$, $S_R \trianglelefteq R(C)$ (of left and right translations) such that:*

1. *$(\forall x, y \in C)$ $x\mathcal{L}y$ in C if and only if $x\mathcal{L}y$ in $(T_L, \underline{C})$;*
2. *$(\forall x, y \in C)$ $x\mathcal{R}y$ in C if and only if $x\mathcal{R}y$ in $(\underline{C}, R_S)$;*
3. *$(T, X, S) \sim (T_L, \underline{C}, S_R)$.*

Conversely, any semigroup biact of this form is faithful, stable, and $\mathcal{J}$-simple.

Proof. Assume that$\mathbf{X} = (T, X, S)$ is faithful, stable, and $\mathcal{J}$-simple. Then $(T, X, S) \sim (T_I, I \times \underline{G} \times \Lambda, S_\Lambda)$ where $(T_I, I \times \underline{G})$ is $\mathcal{L}$-simple and $(\underline{G} \times \Lambda, S_\Lambda)$ is $\mathcal{R}$-simple. Define $C = \mathcal{M}(I, G, \Lambda, P)$ with sandwich matrix $P = (1)$. Then T_I defines a set T_L of left translations of C by $t(i, g, \lambda) = (t.(i, g), \lambda)$ since $t[(i, g, \lambda)(j, h, \mu)] = t(i, gh, \mu) = (t.(i, gh), \mu) = (t.(i, g), \lambda)(j, h, \mu)$ since T acts on $I \times \underline{G}$ by G-endomorphims. By definition of the left translation associated to t, $(i, g, \lambda) \,\mathcal{L}\, (j, g\mu)$ in $(T_L, \underline{C})$ if and only if $\lambda = \mu$ and (i, g) and (j, μ) are $\mathcal{L}$-related in $(T_I, I \times \underline{G})$ if and only if $\lambda = \mu$ if and only if $x \,\mathcal{L}\, y$ in C. We conclude by duality.
The proof of the converse is similar. □

We can actually form another embedding, intermediate between $(T_I, I \times G \times \Lambda, S_\Lambda)$ and $\mathbf{FB}(I, G, \Lambda)$, that will prove interesting to iterate the decomposition process (see next section). Write $\phi = (\phi_1, \phi_2)$, where ϕ is the map defined previously. Then $\phi_1(T) = T_I^{\mathcal{T}}$ is a subsemigroup of $\mathcal{T}(I)$ such that $(T_I^{\mathcal{T}}, I)$ is $\mathcal{L}$-simple, and we have $\phi(T) \trianglelefteq T_I^{\mathcal{T}} \ltimes G^I \trianglelefteq \mathcal{T}(I) \ltimes G^I$ so that:

Corollary 4.16. *Let $\boldsymbol{X} = (T, X, S)$ be a faithful, stable, $\mathcal{J}$-simple semigroup biact. Then there exist two sets I, Λ, a group G, a left subact $(T_I^{\mathcal{T}}, I)$ of $(\mathcal{T}(I), I)$ and a right subact $(\Lambda, S_\Lambda^{\mathcal{T}})$ of $(\Lambda, \mathcal{T}^{op}(\Lambda))$ such that:*

1. *$(T_I^{\mathcal{T}}, I)$ is $\mathcal{L}$-simple;*
2. *$(\Lambda, S_\Lambda^{\mathcal{T}})$ is $\mathcal{R}$-simple;*
3. *$(T, X, S) \hookrightarrow (T_I^{\mathcal{T}}, I) \wr (G, \underline{G}, G) \wr (\Lambda, S_\Lambda^{\mathcal{T}})$ with $T \to T_I^{\mathcal{T}}$ onto, $X \to I \times \underline{G} \times \Lambda$ a bijection and $S \to S_\Lambda^{\mathcal{T}}$ onto.*

Conversely, any semigroup biact of this form is faithful, stable, and $\mathcal{J}$-simple.

Proof. We only prove that $(T_I^{\mathcal{T}}, I)$ is faithful and $\mathcal{L}$-simple. As $T_I^{\mathcal{T}}$ are functions on I, the act $(T_I^{\mathcal{T}}, I)$ is clearly faithful. We prove that it is $\mathcal{L}$-simple. Let $i = R_x, j = R_y \in I$. Then there exists $t \in T$ such that $tx = y$ by $\mathcal{L}$-simplicity. Pose $\phi(t) = (r, g)$, $f(x) = (i, h)$ and $f(y) = (j, k)$. Then $(j, k) = f(y) = f(tx) = \phi(t)f(x) = (r, g)(i, h) = (r(i), g(i)h)$ and $r(i) = j$ with $r \in T_I^{\mathcal{T}}$. The left act $(T_I^{\mathcal{T}}, I)$ is $\mathcal{L}$-simple. □

We deduce from Theorems 4.8 and 4.14 a second equivalent characterization of stable, $\mathcal{J}$-simple biacts.

Corollary 4.17. *Let I, Λ be two sets and G a group. Let $T_I \trianglelefteq End(IG)$ (resp. $S_\Lambda \trianglelefteq End^{op}(G\Lambda)$) be a subsemigroup of the endomorphism monoid of the free right G-act over I (resp. of the (opposite of the) endomorphism monoid of the free left G-act over Λ) such that $(\forall i \in I)T_I(i,1) = I \times \underline{G}$ (resp. $(\forall \lambda \in \Lambda)[(1,\lambda)]S_\Lambda = \underline{G} \times \Lambda$). Then the biact $(T_I, I \times \underline{G} \times \Lambda, S_\Lambda)$ is faithful, stable and $\mathcal{J}$-simple.*
Conversely, any faithful, stable, $\mathcal{J}$-simple semigroup biact is isomorphic to a biact of this form.

Proof. The left act $(T_I, I \times \underline{G})$ is $\mathcal{L}$-simple if and only if $(\forall i \in I, \forall g \in \underline{G})T_I(i,g) = I \times \underline{G}$ if and only if $(\forall i \in I)\, T_I(i,1) = I \times \underline{G}$, and dually. □

Such subsemigroups are coined *transitive subsemigroups of* $End(IG)$ in [16].
Finally, Proposition 4.10 allows a description by matrices:

Corollary 4.18. *Let I, Λ be two sets and G a group. Let $T_I \trianglelefteq \mathcal{M}^c_{I,I}(G)$ (resp. $S_\Lambda \trianglelefteq \mathcal{M}^r_{\Lambda,\Lambda}(G)$) be a subsemigroup of the monoid of matrices over $G \cup \{\star\}$ such that for all $i, j \in I$ and all $g \in G$, there exists $M \in T_I, M(i,j) = g$ and dually for all $\lambda, \mu \in \Lambda$ and all $g \in G$, there exists $M \in S_\Lambda, M(\lambda,\mu) = g$. Then the biact $(T_I, \mathcal{M}^s_{I,\Lambda}(G), S_\Lambda)$ is faithful, stable and $\mathcal{J}$-simple.*
Conversely, any faithful, stable, $\mathcal{J}$-simple semigroup biact is isomorphic to a biact of this form.

Example 4.19. Let G be a group and H, K be subgroups of G such that $HK = G$, and consider the monoid biact $\mathbf{X} = (H, \underline{G}, K)$ with multiplication as actions. The actions are free since G is a group hence faithful. Let $g = hk \in G, h \in H, k \in K$. Then $HgK = HhkK = HK = G$ and G is $\mathcal{J}$-simple. Let also $g = h'k' \in G, h' \in H, k' \in K$. Then $g \,\mathcal{L}\, g'$ if and only if $Hg = Hg'$ and it is left stable Moreover we can identify the set Λ of $\mathcal{L}$-classes with the left coset space H/G. Dually it is right stable with $I \sim G\backslash K$. Finally $g \,\mathcal{H}\, 1 \Leftrightarrow g \in H \cap K$ and we may take as Schützenberger group the group $H \cap K$. By Theorem 4.14, $\mathbf{X}$ admits an embedding into the wreath product $(\mathcal{T}(G\backslash K), G\backslash K) \wr (H \cap K, \underline{H \cap K}, H \cap K) \wr (H/G, \mathcal{T}^{op}(H/G))$ (bijective on its second coordinate).
For instance let Z_{15} be a group of order 15 and Z_3 and Z_5 be the Sylow p-groups of Z_{15} with order 3 and 5 respectively. They are normal subgroups so that Z_3Z_5 is a subgroup of Z_{15} (that contains Z_3 and Z_5). By Lagrange Theorem, we get that $Z_3Z_5 = Z_{15}$. Also $Z_3 \cap Z_5 = \{1\}$. The biact (Z_3, Z_{15}, Z_5) (with multiplication as actions) is faithful, $\mathcal{J}$-simple and stable. By Theorem 4.14, it admits an embedding into the wreath product $(\mathcal{T}(\mathbf{3}), \mathbf{3}) \wr (1, \underline{1}, 1) \wr (\mathbf{5}, \mathcal{T}^{op}(\mathbf{5}))$ (where $\boldsymbol{n}$ is a set with n elements). On the other hand, the biact $(Z_{15}, \underline{Z_{15}}, Z_5)$ (with multiplication as actions) admits an embedding into the wreath product $(\mathcal{T}(\mathbf{3}), \mathbf{3}) \wr (Z_5, \underline{Z_5}, Z_5)$.

Example 4.20. Consider the setup of Example 4.19, with G a group and H, K subgroups of G such that $HK = G$. By Corollary 4.17, $\mathbf{X} = (H, \underline{G}, K)$ is isomorphic to $(M, G\backslash K \times \underline{(H\cap K)} \times H/G, N)$ with M a submonoid of $End(G\backslash K(H\cap K))$ and N a submonoid of $End^{op}((H\cap K)H/G)$. Since H and K are groups, so are M and N. Thus M (resp. N) is a transitive subgroup of the automorphism group of the right $(H \cap K)$-act $G\backslash K(H \cap K)$ (resp. of the left $(H \cap K)$-act $((H \cap K)H/G)$.

Example 4.21. Let $S = \mathcal{M}(I, G, \Lambda, P)$ be a completely simple semigroup and define $\mathbf{X} = (S, \underline{S}, S)$. This biact is faithful. We construct

$$\mathbf{Y} = (\mathcal{T}(I), I) \wr (G, \underline{G}, G) \wr (\Lambda, \mathcal{T}^{op}(\Lambda)) = \big(\mathcal{T}(I) \ltimes G^I, I \times \underline{G} \times \Lambda, G^\Lambda \rtimes \mathcal{T}^{op}(\lambda)\big)$$

and define the map $\Phi = (\phi, f, \psi) : \mathbf{X} \to \mathbf{Y}$ as follows:

1. $\phi : S \to \mathcal{T}(I) \ltimes G^I$ maps $t = (j, g, \lambda)$ to the function $i \mapsto (j, gp_{\lambda i})$;
2. $f : \underline{S} \to I \times \underline{G} \times \Lambda$ is the identity function on $\underline{S}$;
3. $\psi : S \to G^\Lambda \rtimes \mathcal{T}^{op}(\lambda)$ maps $s = (i, g, \mu)$ to the function $\lambda \mapsto (p_{\lambda i}g, \mu)$.

Obviously, $\mathbf{X} = (S, \underline{S}, S)$ is also isomorphic to the biact defined by the semigroups of *inner* left and right translations of the completely simple semigroup S acting on $\underline{S}$.

It comes to no surprise that the representation of a completely simple semigroup in Rees matrix form also entails a simple representation of the associated biact in terms of a matrix biact.

Example 4.22. Consider the setting of Example 4.21, with $S = \mathcal{M}(I, G, \Lambda, P)$ a completely simple semigroup and $\mathbf{X} = (S, \underline{S}, S)$. Pose $Y = \mathcal{M}^s_{I,\Lambda}(G)$ (set of matrices indexed by $I \times \Lambda$ with all coefficients $\star$ except a single one in G) and define $S_I = YP$ subsemigroup of $\mathcal{M}^c_{I,I}(G)$, $S_\Lambda = PY$ subsemigroup of $\mathcal{M}^r_{\Lambda,\Lambda}(G)$. Then $\mathbf{X} \sim (YP, Y, PY)$, where $\phi : t \mapsto C_tP$, $f : x \mapsto C_x$ and $\psi : s \mapsto PC_s$, $t, x, s \in S$ with $C_{(i,g,\lambda)}$ the matrix with g in position i, λ and $\star$ otherwise.

4.3 Structure of completely stable semigroup biacts

We now apply the previous results to completely stable semigroup biacts. Let $\mathbf{X} = (T, X, S)$ be a faithful, completely stable semigroup biact. Then by Lemma 3.17 ${}_TX_S$ is the coproduct of its $\mathcal{J}$-classes (in $\boldsymbol{(T, S)}$**-biacts**), which are stable and $\mathcal{J}$-simple. Let Ω denote the set of $\mathcal{J}$-classes of X. For any $\omega = J_x \in \Omega$, (T, ω, S) is a stable, $\mathcal{J}$-simple biact but not faithful in general. By the previous results, there exist (for each $\omega \in \Omega$) two sets I_ω and Λ_ω, a group G_ω and two subsemigroups $T_\omega \trianglelefteq End_{G_\omega}(I_\omega G_\omega)$ and $S_\omega \trianglelefteq End^{op}_{G_\omega}(G_\omega\Lambda_\omega)$ such that (T, ω, S) admits a surjective morphism onto $(T_\omega, I_\omega \times \underline{G_\omega} \times \Lambda_\omega, S_\omega)$ with the central map a bijection, $(T_\omega, I_\omega \times \underline{G_\omega})$ is $\mathcal{L}$-simple and $(\underline{G_\omega} \times \Lambda_\omega, S_\omega)$ is $\mathcal{R}$-simple. Denote this morphism by $\Phi_\omega = (\phi_\omega, f_\omega, \psi_\omega)$. Finally, form the direct product of the previous semigroups $Q = \Pi_{\omega\in\Omega} T_\omega$, subsemigroup of $\Pi_{\omega\in\Omega} End(I_\omega G_\omega)$, and $P = \Pi_{\omega\in\Omega} S_\omega$, subsemigroup of $\Pi_{\omega\in\Omega} End_{G_\omega}(G_\omega \lambda_\omega)$. These semigroups act on $Y = \bigcup_{\omega\in\Omega} I_\omega \times \underline{G_\omega} \times \Lambda_\omega$ by $(t_\omega, \omega \in \Omega) \cdot y = t_{\omega'} y$ for $y \in I_{\omega'} \times \underline{G_{\omega'}} \times \Lambda_{\omega'}$, and dually. Define $\Phi = (\phi, f, \psi) : (T, X, S) \to (Q, Y, P)$ by $\phi : t \mapsto (\phi_\omega(t), \omega \in \Omega)$, $f(x) = f_{\omega'}(x), x \in \omega'$ and $\psi : s \mapsto (\psi_\omega(s), \omega \in \Omega)$. As (T, X, S) is faithful, the morphism Φ is a monomorphism and by construction, $\phi(T)$ is a subdirect product of $Q = \Pi_{\omega\in\Omega} T_\omega$. Finally, we have proved half of the following statement (which converse is routine):

Corollary 4.23. *Let (T, X, S) be a faithful, completely stable semigroup biact. Then there exists a set Ω, and for each $\omega \in \Omega$ two sets I_ω and Λ_ω, a group G_ω and two subsemigroups $T_\omega \trianglelefteq End_{G_\omega}(I_\omega G_\omega)$ and $S_\omega \trianglelefteq End^{op}_{G_\omega}(G_\omega \Lambda_\omega)$ such that:*

- *$(T_\omega, I_\omega \times G_\omega)$ is $\mathcal{L}$-simple;*
- *$(G_\omega \times \Lambda_\omega, S_\omega)$ is $\mathcal{R}$-simple;*
- *(T, X, S) is isomorphic to (Q', Y, P') with $Y = \bigcup_{\omega\in\Omega} I_\omega \times \underline{G_\omega} \times \Lambda_\omega$, Q' is a subdirect product of $Q = \Pi_{\omega\in\Omega} T_\omega$ and P' is a subdirect product of $P = \Pi_{\omega\in\Omega} S_\omega$.*

Conversely, any semigroup biact (Q', Y, P') of this form is a faithful, completely stable semigroup biact.

The matrix representation appears to be very convenient for the study of coproducts.

Corollary 4.24. *Let I_ω, Λ_ω be sets and G_ω groups indexed by $\omega \in \Omega$, and let I, Λ and G denote their respective disjoint unions over Ω. Let $T_I \trianglelefteq \mathcal{M}^{c,block}_{I,I}(G)$ (resp. $S_\Lambda \trianglelefteq \mathcal{M}^{r,block}_{\Lambda,\Lambda}(G)$) be a subsemigroup of the semigroup of block diagonal matrices with respect to the decomposition $I = \bigcup_{\omega\in\Omega} I_\omega$ with coefficients in $G_\omega \cup \{\star\}$ respectively for each block such that for all $i, j \in I_\omega$, $g \in G_\omega$ there exists $M \in \mathcal{M}^{c,block}_{I,I}(G)$ such that $M(i, j) = g$ (resp. for all $\lambda, \mu \in \Lambda_\omega$, $g \in G_\omega$ there exists $M \in \mathcal{M}^{r,block}_{\Lambda,\Lambda}(G)$ such that $M(\lambda, \mu) = g$). Then the biact $(T_I, \mathcal{M}^{s,block}_{I,\Lambda}(G), S_\Lambda)$ is faithful and completely stable, where $\mathcal{M}^{s,block}_{I,\Lambda}(G)$ is the set of matrices indexed by $I \times \Lambda$ with all coefficients $\star$ except a single one in G_ω on some row $i \in I_\omega$ and column $\lambda \in \Lambda_\omega$, for some $\omega \in \Omega$.*
Conversely, any faithful completely stable biact is isomorphic to a biact of this form.

Example 4.25. Let C be a completely regular semigroup. Then it is a semilattice E of its $\mathcal{J}$-classes J_e, which are completely simple semigroups. Let $T = \Pi_{e\in E} L(J_e)$ direct product of left translations of J_e, and dually $S = \Pi_{e\in E} R(J_e)$. Then $(T, \underline{C}, S)$ is a faithful, completely stable semigroup biact. The matrix representation of each completely simple $\mathcal{J}$-classes J_e entails the previous descriptions.

5 Application - Decomposition of certain completely stable left acts

We first consider the subclass of $\mathcal{L}$-simple left acts, and give structure theorems for these left acts. Then we produce a second structure theorem for completely stable left acts.

5.1 Decomposition of $\mathcal{L}$-simple left acts

Let (T, X) be a faithful, $\mathcal{L}$-simple left semigroup act (one also says that the (left) action of T on X is transitive), and consider any subgroup $G \trianglelefteq Aut^{op}({}_T X)$, automorphism monoid of the T-act ${}_T X$ (with opposite composition as product). First, we recover Oehmke and Steinberg's results by studying the biact (T, X, G)).

Lemma 5.1. *Let (T, X) be a faithful $\mathcal{L}$-simple left monoid act and let G be any subgroup of $Aut^{op}({}_T X)$. Then X carries a (T, G)-biact structure such that X is $\mathcal{J}$-simple, I set of $\mathcal{R}$-classes is equipotent to $(X \backslash G)$, space of right cosets, Λ set of $\mathcal{L}$-classes is a one element set, and we can identify any Schützenberger group with G.*

Proof. By definition of G, X carries a (T, G)-biact structure. As X is $\mathcal{L}$-simple, it is left stable. It is right stable since G is a group (hence $[x]gg^{-1} = x$ for all $x \in X, g \in G$). It is $\mathcal{J}$-simple since it is $\mathcal{L}$-simple by assumption. Still by $\mathcal{L}$-simplicity, $\mathcal{R}$-classes and $\mathcal{H}$-classes coincide, and since G is a group $H_x = xG$ for all $x \in X$, so that I is equipotent to $(X \backslash G)$, space of right cosets. Also we can identify any Schützenberger group with G. More precisely, $\Delta_r(L) = G$, where $L = X$ is the only $\mathcal{L}$-class of X. □

Then, as direct consequences of Corollary 4.18 and Theorem 4.14 we deduce the following lemmas:

Corollary 5.2. *Let (T, X) be a faithful $\mathcal{L}$-simple left semigroup act and let G be any subgroup of $Aut^{op}({}_T X)$. Then $\boldsymbol{X} = (T, X)$ admits an embedding into $(\mathcal{M}^c_{I,I}(G), \mathcal{M}^s_{I,1}(G))$.*

Thus we realized (T, X) as a semigroup of column-monomial matrices over a group G acting on a "vector space" of monomial vectors over G, which is exactly the purpose of Oehmke's main Theorem[20]. But our corollary shows that the assumption of Oehmke that T has a left simple left ideal with idempotent is superflous.

Corollary 5.3. *Let (T, X) be a faithful $\mathcal{L}$-simple left semigroup act and let G be any subgroup of $Aut^{op}({}_T X)$. Then $\boldsymbol{X} = (T, X)$ admits an embedding into the wreath product $(\mathcal{T}(X \backslash G), (X \backslash G)) \wr (G, G)$ bijective on its second coordinate (equivalently, into the left act $(End((X \backslash G)G), (X \backslash G) \times G)$, where $(X \backslash G)G = ((X \backslash G) \times G, G)$ is the free right G-act with basis $(X \backslash G)$).*

This is an analogue of the Kaloujnine-Krasner Theorem[12]. Actually, this extension to monoid acts has already been obtained directly in a very close formulation by Steinberg[27] Corollary 3.17 (for right monoid transitive actions).

We deduce the following simplification for commutative semigroups (see also [5] Lemmas 9, 10 and Theorem 11):

Corollary 5.4. *Let (T, X) be a faithful $\mathcal{L}$-simple left monoid act, with T commutative semigroup. Then T is a group and $(T, X) \sim (T, \underline{T})$.*

Proof. Let (T, X) be a faithful $\mathcal{L}$-simple left monoid act, with T commutative semigroup, and pose $G = Aut^{op}({}_T X)$. We can identify T with a subsemigroup of G ($T \trianglelefteq G$) since T is commutative. Thus (T, X, G) is $\mathcal{R}$-simple and X consists of a single $\mathcal{H}$-class, so that $X \sim \underline{G}$ by Lemma 5.1. Let $x \in X$. We have $G = \Delta_r(L_x) \sim \Delta_l(R_x) \sim stab_l(H_x)$. We prove that $\varepsilon : stab_l(H_x) \to T$, $\delta_t \mapsto t$ is well defined and an isomorphism. Let $t, t' \in T$ such that $\delta_t : H_x \to H_x$ and $\delta_{t'} : H_x \to H_x$ are equal, and let $y \in X$. As $y \,\mathcal{L}\, x$ then exists $s \in T, y = sx$ so that $ty = tsx = stx = st'x = t'sx = t'y$, and t and t' are equal by faithfulness. Thus ε is well defined and as $\delta_{ss'} = \delta_s \delta'_s$ for any $s, s \in T$ it is a morphism. Conversely, let $t \in T$ and $x \in X$. As (T, X) is $\mathcal{L}$-simple then $tx \,\mathcal{L}\, x$ and exists $r \in T$, $rtx = x$ so that $tx = [x]g_t, x = [tx]g_r$ where g_t and $g_r \in G$ are the automorphisms induced by t and r. Finally $t \in stab_l(H_x)$ and the map $d; t \mapsto \delta_t$ is a well defined morphism from T to $stab_l(H_x)$. By definition the maps ε and d are reciprocal, hence isomorphisms. □

Example 5.5. Let $A = (]0, 1], .)$ acting on $\underline{A} =]0, 1[$ by $\alpha(a, y) = y^a (\forall a \in A, y \in \underline{A})$ and form the semidirect product $A \ltimes A$ with product $(a, b)(a', b') = (aa', b(b')^a)$. Let $(U, .)$ be the group of units of $(\mathbb{C}, .)$ viewed as a multiplicative monoid, and $D^* = \{z \in \underline{\mathbb{C}}, 0 < |z| < 1\}$ the open unit disk of $\underline{\mathbb{C}^*}$. Then $T = (A \ltimes A) \times U$ acts faithfully on D^* on the left by $(a, b, \eta) \cdot z = b|z|^{a-1} z\eta$ (for any $(a, b, \eta) \in T$ and $z \in D^*$). Some calculations prove that (T, D^*) is $\mathcal{L}$-simple: let $z = re^{i\theta}, z' = r'e^{i\theta'} \in D^*$. As $r' < 1$ and $0 < r$ then $\lim_{a \to 0} r^a = 1 > r'$ and exists $a \in]0, 1[, r^a \geq r'$. Choose such an element a, and define $b = r'/r^a$. Let finally $\eta = e^{i(\theta' - \theta)}$. Then $(a, b, \eta) \cdot z = b|z|^{a-1} z\eta = br^a e^{i\theta}\eta = r'e^{i\theta'} = z$.
We now find a decomposition as in Corollary 5.3. First, the group U acts on D^* on the right by multiplication, and this action is compatible with the action of T. Indeed, $((a, b, \eta).z) \odot \zeta = b|z|^{a-1} z\eta\zeta = b|z|^{a-1}|\zeta|^{a-1} z\zeta = ((a, b, \eta).(z \odot \zeta)$ since $|\zeta| = 1$. It follows that U is a subgroup of $Aut^{op}({}_T D^*)$. Second, it holds that $D^* =]0, 1[\times \underline{U}$, and the elements of T define endomorphims of the free U-act $]0, 1[U$ since $(a, b, \eta) \cdot (r\zeta\zeta') = br^a \eta\zeta\zeta' = ((a, b, \eta) \cdot (r\zeta))\, \zeta'$ (for any $(a, b, \eta) \in T$ and $0 < r < 1, \eta \in U$). Thus (T, D^*) embeds in $(End(]0, 1[U),]0, 1[\times \underline{U})$ with the monomorphism bijective on its second coordinate.

We finally consider the finite case, and use the embedding (rather than the isomorphism) result (Corollary 4.16) to get an iterated decomposition. First, from Lemma 5.1 and Corollary 4.16 we deduce the following lemma.

Lemma 5.6. *Assume X finite, $|X| = n$. Then either $Aut({}_T X) = \{1_X\}$ or exist a set $I, |I| < n$, and a group $G, |G| \geq 2$ such that $\boldsymbol{X} = (T, X)$ admits an* *embedding bijective on its second coordinate into a wreath product $(T_I, I) \wr (G, G)$ with $T \to T_I$ onto and (T_I, I) a faithful, $\mathcal{L}$-simple left semigroup subact of $(\mathcal{T}(I), I)$.*

As above, we can take $G = Aut^{op}({}_TX)$ as a Schützenberger group of X, and identify I with the space of right cosets $X\backslash G$. By construction, X carries a right action of $G = Aut^{op}({}_TX)$ that is T-equivariant.

By iteration of Lemma 5.6 on faithful, $\mathcal{L}$-simple left semigroup acts until their automorphism group is trivial we get:

Corollary 5.7. *Let (T_0, X_0) be a faithful $\mathcal{L}$-simple semigroup left act with X finite. Then there exist finite sequences:*

- *of non-trivial groups $G_1, \dots, G_p$;*
- *of semigroups $T_1, \dots, T_p$;*
- *of sets $X_1, \dots, X_p$;*

such that for any $k = 1, \dots, p$:

1. *Each group G_k acts on X_{k-1} on the right, and the action is T_{k-1}-equivariant;*
2. *$X_k \sim X_{k-1}\backslash G_k$;*
3. *T_k acts on X_k on the left and each left act (T_k, X_k) is faithful and $\mathcal{L}$-simple;*
4. *(T_{k-1}, X_{k-1}) admits an embedding, bijective on its second coordinate, into the wreath product $(T_k, X_k) \wr (G_k, G_k)$, with $T_{k-1} \to T_k$ onto;*

and such that finally $Aut({}_{T_p}X_p)$ is trivial. Moreover, we can take $G_k \sim Aut^{op}({}_{T_{k-1}}X_{k-1})$ for all $k = 1, \dots, p$.

In particular (T_0, X_0) embeds in the iterated wreath product $(T_p, X_p) \wr (G_p, G_p) \wr (G_{p-1}, G_{p-1}) \wr \dots \wr (G_1, G_1)$. Using the dual construction we get:

Corollary 5.8. *Let (T, X, S) be a faithful $\mathcal{J}$-simple semigroup act with X finite. Then there exist:*

1. *a finite sequence of non-trivial groups G_0, $G^l_1, \dots, G^l_p$, $G^r_1, \dots, G^r_q$;*
2. *a $\mathcal{L}$-simple left act (T_p, X_p) such that $Aut({}_{T_p}X_p)$ is trivial;*
3. *and a $\mathcal{R}$-simple left act (X_q, T_q) such that $Aut^{op}((X_q)_{S_q})$ is trivial*

such that (T, X, S) embeds in the iterated wreath product

$$(T_p, X_p) \wr (G^l_p, G^l_p) \wr \dots \wr (G^l_1, G^l_1) \wr (G_0, G_0, G_0) \wr (G^r_1, G^r_1) \wr \dots \wr (G^r_q, G^r_q) \wr (X_q, S_q)$$

Finally, for a finite, $\mathcal{L}$-simple left semigroup act, it happens that $End({}_TX) = Aut({}_TX)$ (this was recognized in [27] Proposition 2.4, and also appears in Chen's Thesis[4]):

Proposition 5.9. *Let (T, X) be a $\mathcal{L}$-simple left semigroup act with X finite. Then $End(_TX) = Aut(_TX)$.*

Proof. Let $f \in End^{op}(_TX)$. Then $[X]f$ is a T-subact of X and by $\mathcal{L}$-simplicity $[X]f = X$, thus f is surjective. As X is finite then f is bijective, and its reciprocal being an endomorphism, $f \in Aut^{op}(_TX)$. □

5.2 Structure of completely stable left semigroup acts

Regarding completely stable left semigroup acts, we have:

Corollary 5.10. *Let (T, X) be a faithful left semigroup act, completely left stable. Then $(T, X, Aut^{op}(_TX))$ is faithful and completely stable.*

In particular, it decomposes as in Corollaries 4.23 or 4.24.

Proof. By the previous results, $(T, X, Aut^{op}((_TX))$ is a faithful completely left stable biact. Let $x \in X$, $g \in Aut^{op}(_TX)$. Then $[x]gg^{-1} = x$ and $[x]g\,\mathcal{R}\,x$, so that the biact is completely right stable. □

Example 5.11. We consider the semigroup $T = (A \ltimes A) \times U$ of Example 5.5, and define a left action of T on $\mathbb{C}$ by:

$$\begin{cases} (a, b, \eta) \cdot 0 = 0 \\ (a, b, \eta) \cdot z = b|z|^{a-1}z\eta \text{ if } 0 < |z| < 1 \\ (a, b, \eta) \cdot z = z\eta \text{ if } |z| = 1 \\ (a, b, \eta) \cdot z = b^{-1}|z|^{a-1}z\eta \text{ if } 1 < |z| \end{cases}$$

The same calculations as in Example 5.5 show that it is completely left stable, hence the coproduct of its $\mathcal{L}$-classes. There are exactly four $\mathcal{L}$-classes: $\{0\}$, D^*, $\underline{U}$ and $\{z \in \mathbb{C}, |z| > 1\}$. As $(T, \mathbb{C})$ is also faithful, it decomposes as in Corollary 4.23: $(T, \mathbb{C})$ is isomorphic to (T', Y) with

$$Y = \{0\} \cup (]0, 1[\times \underline{U}) \cup \underline{U} \cup (]1, +\infty[\times \underline{U})$$

and T' a subsemigroup of the semigroup of functions $(\mathbb{C}^{[0,+\infty[}, .)$ with pointwise multiplication that leave invariant each of the previous sets $\{0\}$, $(]0, 1[\times \underline{U})$, $\underline{U}$, $(]1, +\infty[\times \underline{U})$, and whose action is transitive on each set.

6 Conclusion and perspectives

The results obtained in this article plaid in favor of a thorough study of semigroup biacts and Green's relations upon them, as it has been the case for semigroups. There are however at least two major and obvious obstacles to a straightforward rewriting of the existing semigroups results. One is the lack of symmetry between the semigroups and the set on which they act. The second is the absence of "idempotents" in semigroup biacts. This renders for instance constructions defined for regular semigroups (that play a prominent role in the local structure theory of semigroups) not directly attainable.

In forthcoming papers, we will try to pursue this research and study notably minimal subacts ("kernels"), Rees quotients and traces of D-classes (biacts with zero), finitness properties based on Green's relations, and extensions of the Green's relations.

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